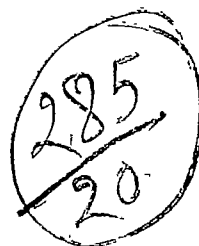
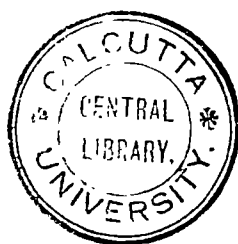


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A PROBLEM ON 'FACTORISATIO NUMERORUM'

BY

D. N. SEN

Introduction

Einar Hille, in Acta Arithmetica, Vol. 2. No. 1 (1936), considers the function $f(n)$, which gives the number of representations of the natural number n as a product of factors greater than one. The object of this paper is to deal with $f(n)$ in greater detail.

Two representations are considered identical, if they contain the same factors in the same order. By definition, $f(1)$ is taken equal to 1.

Let $n = p_{i_1}^{a_1} p_{i_2}^{a_2} \dots p_{i_v}^{a_v}$, where p 's are primes. Clearly, $f(n)$ depends on the divisibility properties of n and not on the values of p 's. Obviously, $f(n)$ is a very irregular function, any of its values being repeated infinitely often.

If $D.d=n$, then there will be $f(d)$ representations of n in which D occurs as the first factor. We have $f(n) = \sum_{d|n}' f(d)$, where the summation extends over all d 's $< n$, $n > 1$.

$$\therefore 2f(n) = \sum_{d|n} f(d), n > 1. \quad (1)$$

By Dedekind's Inversion formula,

$$f(n) = 2f(n) - 2 \sum f\left(\frac{n}{p_{i_1}}\right) + 2 \sum f\left(\frac{n}{p_{i_1} p_{i_2}}\right) - \dots$$

$$\text{or,} \quad f(n) = 2 \left[\sum f\left(\frac{n}{p_{i_1}}\right) - \sum f\left(\frac{n}{p_{i_1} p_{i_2}}\right) + \dots \right] \quad (2)$$

Modification of the formula for $f(n)$, if all a 's be 1

$$\text{Let } N_v = p_1 p_2 \dots p_v.$$

By definition, $f(N_1) = 1$, so that we get from (2),

$$f(N_2) = 2(2-1) = 2$$

$$f(N_3) = 2\{3.2 - 3.1 + 1\} = 8,$$

which are evidently wrong. The reason for this is to be found in the equality (1) which is not true if $n=1$. Hence, in a case in which $f(1)$ appears in (2), a modification is necessary at the end: if v be even, $+1$ is to be added, and if v be odd, then -1 is to be brought in at the end;

$$\text{i.e., } f(N_v) = 2 \left[C_1^v f(N_{v-1}) - C_2^v f(N_{v-2}) + \dots + (-1)^{v-1} C_v^v f(1) \right] + (-1)^v \dots (3)$$

$$\text{This gives } f(N_2) = 2[2.1 - 1.1] + 1 = 3$$

$$f(N_3) = 2[3.3 - 3.1 + 1.1] - 1 = 13$$

$$f(N_4) = 2[4.13 - 6.3 + 4.1 - 1.1] + 1 = 75, \text{ and so on.}$$

A formula for $f(p^a)$

$$\text{From (2), } f(p^a) = 2f(p^{a-1}) = \dots 2^{a-1}f(p) = 2^{a-1}.$$

We can also get the result in the following way, which admits of generalisation.

$$\text{Let } s(x) = f(p^0) + f(p^1)x + f(p^2)x^2 + f(p^3)x^3 + \dots$$

$$\text{Then } 2s(x) - 1 = f(p^0) + 2f(p^1)x + 2f(p^2)x^2 + 2f(p^3)x^3 + \dots$$

$$(1-x)\{2s(x) - 1\} = f(p^0) + f(p^1)x + f(p^2)x^2 + f(p^3)x^3 + \dots$$

$$\therefore 2f(p^a) - 2f(p^{a-1}) = f(p^a), a \geq 2$$

$$= s(x).$$

$$\therefore s(x) = \frac{1-x}{2(1-x)-1} \dots (4)$$

$$\text{i.e., } f(p^a) \text{ is the coefficient of } x^a \text{ in } (1-x)(1-2x)^{-1} \dots (5)$$

$$= (1-x)(1+2x+2^2x^2+2^3x^3+\dots+2^{a-1}x^{a-1}+2^ax^a+\dots), 0 < x < \frac{1}{2}$$

$$\text{or, } f(p^a) = 2^a - 2^{a-1} = 2^{a-1}.$$

General value of $f(p^a q^\beta)$, ($a > \beta$)

$$f(p^a q^\beta) = \frac{1}{2} \cdot \frac{1}{[a][\beta]} \left\{ \frac{\partial^a}{\partial x_0^a} \cdot \frac{\partial^\beta}{\partial y_0^\beta} \cdot \frac{1}{2(1-x)(1-y)-1} \right\},$$

(the suffix zero being used to indicate substitution of zero after differentiation)

$$\begin{aligned} &= \frac{1}{2} \cdot \frac{1}{[a][\beta]} \cdot \frac{\partial^\beta}{\partial y_0^\beta} \cdot \frac{z^a(1-y)^a \cdot a!}{\{2(1-y)-1\}^{a+1}} \\ &= \frac{1}{2} \cdot \frac{2^a}{[a][\beta]} \cdot [a+\beta \cdot 2^\beta - \beta C_1 \cdot (a) (\underline{a+\beta-1 \cdot 2^{\beta-1}}) \\ &\quad + \beta C_2 \cdot a(a-1) (\underline{a+\beta-2 \cdot 2^{\beta-2}}) + \dots \\ &\quad + (-1)^\beta \beta C_\beta \cdot a(a-1) \dots (a-\beta+1) (\underline{a})], \end{aligned}$$

by Leibnitz's theorem

$$\begin{aligned} &= \frac{2^{a-1} [a+\beta]}{[a][\beta]} \left[2^\beta - \frac{\beta \cdot a}{(a+\beta) \cdot 1} 2^{\beta-1} + \frac{\beta(\beta-1)a(a-1)}{(a+\beta)(a+\beta-1) \cdot 1 \cdot 2} 2^{\beta-2} - \right. \\ &\quad \left. \dots + (-1)^\beta \cdot \frac{\beta(\beta-1) \dots 2 \cdot 1 \cdot a(a-1) \dots (a-\beta+1)^i}{(a+\beta)(a+\beta-1) \dots (a+1) \cdot 1 \cdot 2 \cdot 3 \dots \beta} \right] \\ &= \frac{2^{a+\beta} [a+\beta]}{[a][\beta]} \cdot F(-a, -\beta, -a+\beta, \frac{1}{2}), \end{aligned} \quad \dots (9)$$

where F denotes the usual hypergeometric series.

Another form of $f(p^a q^\beta)$

$$\begin{aligned} f(p^a q^\beta) &= \frac{1}{2} \cdot \frac{1}{[a][\beta]} \cdot \frac{\partial^\beta}{\partial y_0^\beta} \cdot \frac{\{2(1-y)\}^a \cdot a!}{\{2(1-y)-1\}^{a+1}} \\ &= \frac{1}{2[\beta]} \cdot \frac{\partial^\beta}{\partial y_0^\beta} \cdot \left[\frac{1}{2(1-y)-1} + {}^a C_1 \cdot \frac{1}{\{2(1-y)-1\}^2} \right. \\ &\quad \left. + {}^a C_2 \cdot \frac{1}{\{2(1-y)-1\}^3} + \dots + {}^a C_a \cdot \frac{1}{\{2(1-y)-1\}^{a+1}} \right] \\ &= \frac{1}{2[\beta]} \cdot 2^\beta \left\{ \beta! + {}^a C_1 \cdot \frac{(\beta+1)!}{1!} + {}^a C_2 \cdot \frac{(\beta+2)!}{2!} + \dots + {}^a C_a \cdot \frac{(a+\beta)!}{a!} \right\} \end{aligned}$$

$$\begin{aligned}
 &= \frac{2^{\beta-1}}{|\beta|} \cdot \beta! \left\{ 1 + \frac{\alpha \cdot (\beta+1)}{1.1} + \frac{\alpha(\alpha-1)}{1.2} \frac{(\beta+1)(\beta+2)}{1.2} + \dots \right. \\
 &\quad \left. + \frac{\alpha(\alpha-1)\dots 1}{1.2\dots\alpha} \cdot \frac{(\beta+1)(\beta+2)\dots(\beta+\alpha)}{1.2\dots\alpha} \right\} \\
 &= 2^{\beta-1} F(-\alpha, \beta+1, 1, -1). \quad \dots (10)
 \end{aligned}$$

By symmetry,

$$f(p^\alpha q^\beta) = 2^{\alpha-1} F(-\beta, \alpha+1, 1, -1). \quad \dots (11)$$

A recurrence-formula for $f(N_v)$

The total number in $f(N_v)$ may be considered, first

(i) by taking any factor in the first position and then associating with it the $f(N_{v-1})$ possible arrangements with the other factors; then

(ii) by taking a product of 2 factors for the first position and associating with it $f(N_{v-2})$ possible arrangements with the other factors, and so on.

This gives

$$f(N_v) = {}^v C_1 \cdot f(N_{v-1}) + {}^v C_2 \cdot f(N_{v-2}) + \dots + {}^v C_v f(1).$$

By relation (3),

$$\begin{aligned}
 f(N_v) &= 2 {}^v C_1 \cdot f(N_{v-1}) - 2 {}^v C_2 f(N_{v-2}) + \dots \\
 &\quad + 2(-1)^{v-1} \cdot {}^v C_v \cdot f(1) + (-1)^v.
 \end{aligned}$$

These lead to the new recurrence relation

$$f(N_v) = 4 \left\{ {}^v C_2 \cdot f(N_{v-2}) + {}^v C_4 \cdot f(N_{v-4}) + \dots \right\} - (-1)^v. \quad \dots (12)$$

A formula for $f(N_r)$

From (8),

$$f(N_r) = \frac{1}{2} \cdot \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} \dots \frac{\partial}{\partial x_r} \left[\frac{1}{2(1-x_1)(1-x_2) \dots (1-x_r) - 1} \right]_0$$

$$\therefore f(N_1) = \frac{1}{2} \cdot \frac{\partial}{\partial x_0} \left[\frac{1}{2(1-x)-1} \right] = \frac{1}{2} \cdot \left[\frac{2}{\{2(1-x)-1\}^2} \right]_0 = 1.$$

$$\begin{aligned} f(N_2) &= \frac{1}{2} \cdot \frac{\partial}{\partial x_0} \frac{\partial}{\partial y_0} \left[\frac{1}{2(1-x)(1-y)-1} \right] = \frac{1}{2} \frac{\partial}{\partial y_0} \left[\frac{2(1-y)}{\{2(1-y)-1\}^2} \right] \\ &= \frac{1}{2} \cdot \frac{\partial}{\partial y_0} \left[\frac{1}{2(1-y)-1} + \frac{1}{\{2(1-y)-1\}^2} \right] \\ &= \frac{1}{2} [2 + 2 \cdot 2] = 3. \end{aligned}$$

$$\begin{aligned} f(N_3) &= \frac{1}{2} \cdot \frac{\partial}{\partial z_0} \frac{\partial}{\partial y_0} \frac{\partial}{\partial x_0} \left[\frac{1}{2(1-x)(1-y)(1-z)-1} \right] \\ &= \frac{1}{2} \frac{\partial}{\partial z_0} \cdot \frac{\partial}{\partial y_0} \cdot \left[\frac{1}{2(1-y)(1-z)-1} + \frac{1}{\{2(1-y)(1-z)-1\}^2} \right] \\ &= \frac{1}{2} \cdot \frac{\partial}{\partial z_0} \left[\frac{2(1-z)}{\{2(1-z)-1\}^2} + \frac{2! \cdot 2(1-z)}{\{2(1-z)-1\}^3} \right] \\ &= \frac{1}{2} \cdot \frac{\partial}{\partial z_0} \left[\frac{1}{2(1-z)-1} + \frac{1}{\{2(1-z)-1\}^2} \right. \\ &\quad \left. + 2! \left\{ \frac{1}{(2\bar{1}-z-1)^2} + \frac{1}{(2\bar{1}-z-1)^3} \right\} \right] \\ &= \frac{1}{2} [2 + 2 \cdot 2 + 2! \{2 \cdot 2 + 3 \cdot 2\}] \\ &= \frac{1}{2} \cdot 26 = 13. \end{aligned}$$

From the successive generations we can arrange the terms in the following way :

$2f(N_1)$	...	1	1			
$2f(N_2)$	...	1	3	2		
$2f(N_3)$	...	1	7	12	6	
$2f(N_4)$	...	1	15	50	60	24

.....

The general scheme for the formation of terms will be

$$u_{n,m} = mu_{n-1,m} + (m-1)u_{n-1,m-1}, \quad \dots \quad (13)$$

where the first suffix refers to the row and the second one to the column. It can be seen, by induction, that the elements are

$$\begin{array}{lll} 1^1, & 2^1 - 1^1 & \\ 1^2, & 2^2 - 1^2, & 3^2 - 2.2^2 + 1^2 \\ 1^3, & 2^3 - 1^3, & 3^3 - 2.2^3 + 1^3, & 4^3 - 3.3^3 + 3.2^3 - 1^3 \\ 1^4, & 2^4 - 1^4, & 3^4 - 2.2^4 + 1^4, & 4^4 - 3.3^4 + 3.2^4 - 1^4, \\ & & 5^4 - 4.4^4 + 6.3^4 - 4.2^4 + 1^4, \\ \text{etc.} & \text{etc.} & \text{etc.} \end{array}$$

so that

$$\begin{aligned} u_{n,m} &= m^n - m^{n-1}C_1(m-1)^n + m^{n-2}C_2(m-2)^n - \dots \text{ to } m \text{ terms} \\ &= \text{coeff. of } x^n \text{ in } \underline{n}.e^{mx}(1-e^{-x})^{m-1} \\ &= \quad \quad \quad \underline{n}.(e^x-1)^{m-1}.e^x. \end{aligned}$$

$$\begin{aligned} \therefore u_{n,1} + u_{n,2} + \dots + u_{n,m} + \dots + u_{n,n+1} \\ &= \text{coeff. of } x^n \text{ in } \underline{n}.e^x[1 + (e^x-1) + (e^x-1)^2 + \dots + (e^x-1)^n] \\ &= \quad \quad \quad \underline{n}.e^x \cdot \frac{[(e^x-1)^{n+1} - 1]}{e^x - 2} \\ &= \quad \quad \quad \underline{n}. \frac{e^x}{2 - e^x}, \end{aligned}$$

as in the first part, the least power of x is x^{n+1} . (Evidently, suitable values of x may be taken to ensure validity of the expansions involved.)

$$\begin{aligned} \text{Thus} \quad 2f(N_n) &= \text{coeff. of } x^n \text{ in } \underline{n} \frac{e^x}{2 - e^x} \\ &= \quad \quad \quad \underline{n} \left\{ \frac{e^x}{2 - e^x} + 1 \right\}. \\ &= \quad \quad \quad \underline{n} \frac{2}{2 - e^x} \end{aligned}$$

$$\begin{aligned}
 &= \text{coeff. of } x^n \text{ in } \left(1 - \frac{e^x}{2}\right)^{-1} \\
 &= \frac{1^n}{2} + \frac{2^n}{2^2} + \frac{3^n}{2^3} + \dots
 \end{aligned}$$

Thus $f(N_n) = \frac{1}{2} \sum_{k=1}^n \frac{k^n}{2^k} \dots \quad (14)$

SCIENCE COLLEGE,
PATNA,

ON THE APPLICATION OF ELLIPTIC FUNCTIONS TO TWISTED QUARTIC

By

ASGHAR HAMEED

(Communicated by Prof. C. V. H. Rao)

*Introduction **

In the first article we apply elliptic functions to prove a result, believed to be new, in regard to the plane cubic. This is later generalised to the case of the twisted quartic (§5). We show a connection between the set of eight associated points in space and the set of nine associated points in a plane (§7). This is followed by a discussion on the stationary points of the twisted quartic (§§8-11). We also consider certain closed groups of points on the quartic (§12).

Indebtedness is due to Professor C. V. Hanumanta Rao for his valuable advice and help.

1. If a conic be drawn through four fixed points of a plane elliptic cubic, then the line joining the remaining two points common to the conic and the cubic passes through a fixed point† of the cubic. If u_1, u_2, u_3 and u_4 be the parametric representations of the four points respectively, then the parametric representation w'_5 of the last fixed point is equal to $(u_1 + u_2 + u_3 + u_4)$. Thus starting with any five points u_1, u_2, u_3, u_4 and u_5 of the cubic we obtain five other points w'_1, w'_2, w'_3, w'_4 and w'_5 , also lying on the cubic. We now state and prove that *the necessary and sufficient condition that the conics determined by the sets of points $u_1, \dots, u_5$ and $w'_1, \dots, w'_5$ should intersect in a point on the cubic is that the sixth point of intersection with the cubic of either conic be a point of inflexion; when the condition is satisfied, then the other conic also passes through the same point of inflexion, and the relation between the two sets of points is mutual.*

* This paper was read before the Second Subject Conference in Mathematics of the Punjab University, on March 29, 1941.

† Baker, Principles of Geometry, Vol. III, Ex. 2, p. 178.

For, by hypothesis,

$$u'_1 \equiv u_2 + u_3 + u_4 + u_5 \equiv \Sigma u_1 - u_1, \text{ etc.} \quad \dots (i)$$

$$\text{Hence} \quad \Sigma u'_1 \equiv 5\Sigma u_1 - \Sigma u_1 \equiv 4\Sigma u_1 \quad \dots (ii)$$

Now let the conic $u_1 u_2 u_3 u_4 u_5$ meet the cubic further in u , and suppose the other conic also passes through u . Then,

$$\Sigma u_1 + u \equiv 0 \pmod{\text{period}}$$

and

$$\Sigma u'_1 + u \equiv 0 \pmod{\text{per.}}$$

Hence

$$\Sigma u_1 \equiv \Sigma u'_1 \pmod{\text{per.}}$$

Thus from (ii), $3\Sigma u_1 \equiv 0 \pmod{\text{per.}}$, which shows that $u (\equiv -\Sigma u_1)$ is a point of inflexion of the cubic.

Hence the condition is necessary. To show that it is also sufficient, let the conic $u_1 \dots u_5$ meet the cubic further in a point of inflexion $\frac{P}{3}$, where P is any period.

$$\text{Then we have} \quad \Sigma u_1 + \frac{P}{3} \equiv 0 \pmod{\text{per.}} \quad \dots (iii)$$

$$\text{Multiplying by four, we get} \quad 4\Sigma u_1 + \frac{P}{3} \equiv 0 \pmod{\text{per.}}$$

Thus from (ii), we get $\Sigma u'_1 + \frac{P}{3} \equiv 0 \pmod{\text{per.}}$, showing that the second conic passes through the same point of inflexion. Hence the condition is also sufficient.

It remains to be shown that when the condition is satisfied, the relation between the two sets of points is mutual, *i.e.*, just as the first set gives rise to the second set, so also the second gives rise to the first. For this, it is enough to show that $u_1 \equiv \Sigma u'_1 - u'_1$. But this is true, for,

$$\begin{aligned} \Sigma u'_1 - u'_1 &\equiv 3\Sigma u_1 + u_1, \quad \text{from (i) and (ii)} \\ &\equiv u_1 \quad \text{from (iii).} \end{aligned}$$

2. Let us now consider the quartic curve τ^* given by the intersection of two quadrics. It may be represented parametrically by the following equations:

$$x = p^2(u), \quad y = p(u), \quad z = 1, \quad t = p'(u),$$

* The contents of this article are taken from a lecture by Professor C. V. Hanumanta Rao.

where $p(u)$ is the Weierstrassian Elliptic Function. Thus the parameters of the $4n$ points in which τ meets a surface of the n th degree, add up to a period. In particular the sum of the parameters of eight associated points on τ is a period.

If the osculating plane at a point P of τ meets τ further in the point P' , then P' may be said to be the *Post-osculation* of P , and P may be said to be a *Pre-osculation* of P' . The algebraic relation satisfying P and P' is $P' + 3P \equiv 0 \pmod{\text{per.}}$. This shows that a point on τ has just one post-osculation, but nine pre-osculations. If the point P' coincides with P , then P may be said to be a *stationary point* of τ , and the osculating plane at P , a *stationary plane* of τ . Evidently P must satisfy the algebraic relation $4P \equiv 0 \pmod{\text{per.}}$, showing that τ has sixteen stationary points. By fours, the sixteen stationary planes of τ envelop the four cones which pass through it.

3. Let us find out the points of τ lying in the fundamental planes $x=0$, $y=0$, $z=0$ and $t=0$. The general point on τ may be put down as $(p^2(u), p(u), 1, p'(u))$.

If C be a root of $p(u)=0$, then the four points in which τ is met by the plane $x=0$, are $\pm C$, each taken twice. Thus $x=0$ touches τ at the two points $\pm C$, the join of which is therefore a generator of one of the four cones which pass through τ .

To find the intersections of τ with $y=0$, either the y -coordinate in $(p^2(u), p(u), 1, p'(u))$ must be zero, or some other coordinate be ∞ . The former case gives us $u \equiv \pm C$, and the latter gives us $u=0$ twice. Hence the plane $y=0$ meets τ at the points $\pm C$ and touches it at the origin.

Similarly, on dividing by $p^2(u)$ throughout, we see that $z=0$ meets τ in the four points which coincide with the origin. And the plane

$t=0$ meets τ in the points $0, \frac{\omega}{2}, \frac{\omega'}{2}$ and $\frac{\omega+\omega'}{2}$ of which the first is

obtained on division throughout by $p^2(u)$ and the remaining three are the roots of $p'(u)=0$.

It appears then that the four fundamental planes meet τ in only six distinct points, namely, $0, \frac{\omega}{2}, \frac{\omega'}{2}, \frac{\omega+\omega'}{2}, C, -C$. Of these, the first four are stationary points and they lie in the same plane $t=0$.

* ω and ω' are the periods of $p(u)$.

It may also be observed that $z=0$ is a stationary plane; the edges $x=y=0$ and $y=z=0$ are bisecants of τ , the latter being the tangent line at the origin; each of the two edges $y=t=0$, and $z=t=0$ meets τ in one point; and the edges $x=z=0$, and $x=t=0$ do not meet τ .

4. Now, if V be the vertex of a cone Q of degree n , and r be a curve of order k , and passing through V , then apart from V , the curve meets the cone in $(nk-n)$ points. For, the curve r projects from V on an arbitrary plane in a curve r' of order $(k-1)$. The section of the cone by the plane is a curve of order n . These two curves intersect in $n(k-1)$ points which are the projections of the common points of Q and r , apart from V . In particular, if the vertex V of a quadric cone lies on τ , it will meet τ in six other points.

5. We shall now generalize the results of §1. Let V , u_1 , u_2 , u_3 and u_4 be fixed points of τ . Then we have the following:—

If a quadric cone with V as vertex be drawn to pass through the fixed points $u_1, \dots, u_4$ of τ , then the line joining the remaining two points common to the cone and the quartic meets a fixed bisecant of τ which passes through V . In other words that line is a generator of one system of a fixed quadric through τ .

For, if the remaining two common points be u and u' , we have

$$2V + u_1 + u_2 + u_3 + u_4 + u + u' \equiv 0 \text{ (mod. per.)}$$

which shows that the line joining u and u' is coplanar with, and thus meets the line joining V and $(V + u_1 + u_2 + u_3 + u_4)$. Let this last point be called u'_5 . Thus starting with V and a set of five points $u_1, \dots, u_5$, all lying on τ , we can construct another set of five points $u'_1, \dots, u'_5$, also lying on τ . We now state the following:—

The necessary and sufficient condition that the two cones having V as a common vertex, and passing respectively through the sets of points $u_1, \dots, u_5$ and $u'_1, \dots, u'_5$ should intersect in a point on the quartic is that the sixth point of intersection with the quartic of either cone be a pre-osculation of V ; when this condition is satisfied, the other cone also passes through the same pre-osculation, and the relation between the two sets of points is mutual.

This is seen to be true by §1, for if we project τ on an arbitrary plane, from V , then we get a plane cubic whose nine points of inflexion arise as the projections of the nine pre-osculationals of V . The theorem may, however, be proved by the help of elliptic functions.

As the procedure is similar to that of §1, we shall here prove only the last part, *i.e.*, when the condition is satisfied, then the set $u'_1, \dots, u'_5$ gives rise to the set $u_1, \dots, u_5$ just as the latter set gave rise to the former set. It is therefore enough to show that

$$u_1 \equiv V + u'_2 + u'_3 + u'_4 + u'_5.$$

By hypothesis,

$$u'_1 \equiv V + \Sigma u_1 - u_1. \quad \dots (i)$$

$$\text{Hence} \quad \Sigma u'_1 \equiv 5V + 4\Sigma u_1. \quad \dots (ii)$$

Also, since the condition is satisfied, we have

$$2V + \Sigma u_1 + \left(-\frac{1}{3}V + \frac{P}{9}\right) \equiv 0 \pmod{\text{per.}}, \quad \dots (iii)$$

for $\left(-\frac{1}{3}V + \frac{P}{9}\right)$ is the general pre-osculation of V , where P is a period.

Multiplying (iii) by 3, we get

$$5V + 3\Sigma u_1 \equiv 0 \pmod{\text{per.}}. \quad \dots (iv)$$

$$\begin{aligned} \text{Hence,} \quad V + \Sigma u'_1 - u'_1 &\equiv V + 4V + 3\Sigma u_1 + u_1, & \text{from (i) and (ii)} \\ &\equiv u_1 & \text{from (iv),} \end{aligned}$$

which was to be proved.

6. A cubic cone with its vertex V lying on τ will meet the quartic in nine other points, say, $u_1, u_2, \dots, u_9$ (§4). And we have, $\Sigma u_1 + 3V \equiv 0 \pmod{\text{per.}}$. It follows that if any cubic cone with V as vertex, passes through eight of the nine points $u_1, \dots, u_9$, it will also pass through the ninth. The nine lines $Vu_1, Vu_2, \dots, Vu_9$ may thus be said to form a set of nine associated lines. Now an arbitrary plane meets a cubic cone in a cubic curve, and on that plane, the projection of τ from V is also a cubic curve. Hence the set of nine associated lines meet an arbitrary plane in a set of nine associated points. Conversely, by joining the nine points of an associated set in a plane, to a point outside the plane, we get a set of nine associated lines. It also follows that all the cubic cones, which have eight generators in common, have also a ninth common generator namely the generator associated with the original eight.

7. From an arbitrary point in space, sixteen common transversals can be drawn to two arbitrary twisted quartic curves. This may be seen to be true by projecting the two curves from that point on an arbitrary plane. The two plane curves intersect in sixteen points, which, joined to the vertex of projection, give the sixteen common transversals. Similarly we may prove that if two twisted quartic curves have a point V in common, they have nine common bisecants through it.

Now, let $A_1, \dots, A_7$ and V be the eight points in which an arbitrary quadric meets τ . These eight points then form an associated set. On the basis of the last article, we proceed to show a connection between this associated set, and the set of nine associated points in a plane.

Since every quadric through $A_1, \dots, A_7$ must pass through V , it follows that every twisted elliptic quartic through $A_1, \dots, A_7$ must also pass through V . As has been remarked above, such a quartic curve will have nine bisecants through V , in common with τ . Seven of them are $VA_1, \dots, VA_7$. If an eighth, say VA_8 , is fixed, then by the last article, the ninth is thereby uniquely determined. For, the quartic curves in question lie on cubic cones which have eight generators, namely $VA_1, \dots, VA_8$ in common. These cones then have a ninth generator in common which is the ninth common bisecant of the quartic curves. Call it VA_9 . Projecting from V on an arbitrary plane, and denoting the projection of A_1 by A'_1 , etc., it follows that all plane cubics which pass through eight common points $A'_1, \dots, A'_8$, pass also through a ninth common point A'_9 . Thus $A'_1, \dots, A'_9$ form a set of nine associated points in a plane.

8. We have already remarked that if τ be projected on an arbitrary plane, from a point V lying on it, we get an elliptic cubic whose points of inflexion are the projections of the nine pre-osculationals of V in regard to τ (§5). Hence, or by considering the parameters of the nine pre-osculationals of the arbitrary point V on τ , it follows that *they lie on each of four triads of planes concurrent at V , each plane of a triad containing three of the pre-osculationals.*

If, however, V be a stationary point, then one of its pre-osculationals coincides with it. For, if V be the stationary point $\frac{P_1}{4}$ where

P_1 is a period, then its pre-osculationals are given by $\left(-\frac{1}{3} \frac{P_1}{4} + \frac{P}{3}\right)$.

where P is a period. When P is given the value P_1 , the pre-osculation is seen to coincide with $\frac{P_1}{4}$, i.e., V . Also the sum of the parameters of the remaining eight pre-osculationals is

$$\left\{ \left(-9, \frac{1}{3}, \frac{P_1}{4} \right) - \frac{P_1}{4} \right\}, \text{ i.e., } -P_1.$$

Hence we have the result that a stationary point has only eight pre-osculationals, and that these eight points form an associated set.

9. The sixteen stationary points of τ , given by $4u \equiv 0 \pmod{\text{per.}}$, are tabled below :

$\frac{3\omega'}{4}$	$\frac{\omega + 3\omega'}{4}$	$\frac{2\omega + 3\omega'}{4}$	$\frac{3\omega + 3\omega'}{4}$
$\frac{2\omega'}{4}$	$\frac{\omega + 2\omega'}{4}$	$\frac{2\omega + 2\omega'}{4}$	$\frac{3\omega + 2\omega'}{4}$
$\frac{\omega'}{4}$	$\frac{\omega + \omega'}{4}$	$\frac{2\omega + \omega'}{4}$	$\frac{3\omega + \omega'}{4}$
0	$\frac{\omega}{4}$	$\frac{2\omega}{4}$	$\frac{3\omega}{4}$

Since every one of them is one-fourth of a period, the sum of any three of them is also one-fourth of a certain period. So also the negative of this sum. Hence it follows that the plane joining any three of them meets τ again in a stationary point.* But this last stationary point may not be distinct from the first three, e.g., the plane of the points $0, \frac{\omega}{4}$, and $\frac{2\omega}{4}$ meets τ again in $\frac{\omega}{4}$. In §11 we shall enumerate the planes each of which passes through four distinct stationary points, or through three distinct stationary points, touching the quartic at one of them.

* This corresponds to the statement that the line joining two points of inflexion of a plane elliptic cubic meets it again in a point of inflexion.

It may first be observed that with each stationary point we can associate three others, such that the four form the vertices of a parallelogram whose horizontal side is $\frac{\omega}{2}$ and whose inclined side is $\frac{\omega'}{2}$.

The sixteen stationary points can be divided into four such sets given below:

I		II	
$\frac{2\omega'}{4}$	$\frac{2\omega + 2\omega'}{4}$	$\frac{\omega + 2\omega'}{4}$	$\frac{3\omega + 2\omega'}{4}$
0	$\frac{2\omega}{4}$	$\frac{\omega}{4}$	$\frac{3\omega}{4}$
III		IV	
$\frac{3\omega'}{4}$	$\frac{2\omega + 3\omega'}{4}$	$\frac{\omega + 3\omega'}{4}$	$\frac{3\omega + 3\omega'}{4}$
$\frac{\omega'}{4}$	$\frac{2\omega + \omega'}{4}$	$\frac{\omega + \omega'}{4}$	$\frac{3\omega + \omega'}{4}$

The four points of each set add up to a period, showing that they are coplanar. Also, they are halves of the same half-period, which shows* that the tangent lines to τ , at the four stationary points of any one of the four sets, are generators of one of the four cones which pass through τ . Thus the four sets I,..., IV correspond to the four cones which pass through τ .

Now, take any stationary point S and the cone Q corresponding to the set to which S belongs. The tangent plane to Q at S touches τ at each of the two points where τ is met by the generator of Q through S. But, as already remarked, this generator is the tangent line to τ at S. Thus the two points of contact of τ and the tangent plane of Q at S coincide, showing that the tangent plane is the

* This follows from a remark made by Professor Rao, in the lecture referred to above, that the sum of the parameters of the two points in which τ is met by any generator of one system of a quadric, which passes through τ , is a constant C, the corresponding sum for the generators of the other system being $-C$. Thus for a cone through τ , we have $C \equiv -C \pmod{\text{per.}}$, i.e., C must be a half-period.

stationary plane at S . Hence we have proved the statement that the stationary planes at the stationary points of any one of the four sets I, ..., IV, envelop the corresponding cone which passes through the quartic (§ 2).

By the second footnote to this article, it follows that the tangent line to τ at a stationary point and the joins of the point to the remaining three of the set to which it belongs are the generators through that stationary point of the four cones which pass through the quartic.

10. Now, let the line joining two stationary points of the same set be called a *stationary bisecant* of the *first type*, and the line joining two stationary points, belonging to different sets, be called a *stationary bisecant* of the *second type*. There are 24 bisecants of the first type, and 96 of the second type.

We may state that *through a stationary bisecant of the first type can be drawn five planes each of which meets τ in four distinct stationary points, and four planes each of which meets τ in only three distinct stationary points, touching it at one of them*. To verify this, let us consider the stationary bisecant joining 0 and

$\frac{2\omega}{4}$ both of which belong to set I. It may be seen that the plane joining this stationary bisecant to a point of any one of the two sets III and IV meets τ again in another point of the same set. This gives us four planes through the stationary bisecant, each of which meets τ in four distinct stationary points; the fifth plane with this property is given by the four points of set I. The plane joining the stationary bisecant to any point of set II touches τ at that point. This gives us the four planes each of which meets τ in only three distinct stationary points, touching it at one of them. It may be observed that the four points of contact are distinct from the two stationary points whose join has been considered, and belong to the same set.

Similarly, we may state that *through a stationary bisecant of the second type can be drawn six planes each of which meets τ in four distinct stationary points, and two planes each of which meets τ in only three distinct stationary points, touching it at one of them*. To verify this, let us consider the stationary bisecant of the second type which joins 0 and $\frac{\omega}{4}$ which belong respectively to sets I and II.

Omitting these two points and the other two which belong to their

rows in their respective sets, we are left with twelve stationary points which may be seen to group themselves in six pairs such that the two points of a pair are coplanar with the two with which we started. Thus we get the six planes as stated above. And joining the stationary bisecant to each of the two points, omitted above, namely $\frac{2\omega}{4}$ and $\frac{3\omega}{4}$, we get two planes which touch τ at $\frac{\omega}{4}$ and 0 respectively.

It may be observed that, in this case, the points of contact are the same as those whose join has been considered.

11. We have remarked that there are 24 stationary bisecants of the first type and 96 of the second type. Thus, from the two statements of the previous article, it follows that *the number of planes each of which passes through four distinct stationary points is 116* ($= \frac{24 \times 5 + 96 \times 6}{6}$, division by six being due to the fact that

each plane will arise from six stationary bisecants). *Of these, 29 pass through any one of the sixteen stationary points.* For, through any stationary point can be drawn three bisecants of the first type, and twelve of the second type, giving the number of the required planes $= \frac{3 \times 5 + 12 \times 6}{3} = 29$. This also follows from the fact that the solution of $16x = 4 \times 116$ is 29.

Similarly, *the number of planes, each of which passes through three distinct stationary points, touching τ at one of them, is 96* ($= \frac{24 \times 4 + 96 \times 2}{3}$, division by three being due to the fact that each plane will arise from three stationary bisecants). *Of these, 18 pass through any one of the sixteen stationary points.* For, as before, $\frac{3 \times 4 + 12 \times 2}{2} = 18$, the division by two being due to the fact that in the above enumeration, each of the planes comes twice, namely from the two stationary bisecants obtained by joining the stationary point under consideration to the other two stationary points in that plane.

12. We* have already defined the pre-osculation and the post-osculation of a point on the quartic τ (§ 2). More generally, by a

* In this article we shall generalize to the twisted quartic some of the results given by Charles Tweedie. The Collinearities of the Points of Inflection of a Non-Singular Plane Cubic and their Pretangentials, Rendiconti del Circolo Matematico di Palermo, t. XXIX, 1° Sémestre, 1910, pp. 347-53.

pre-osculation of the m th order of a point P on τ is meant a point whose m th post-osculation taken consecutively is P .

When P is a stationary point of the quartic, the peculiarity arises that it may be considered as a pre-osculation or post-osculation of the m th order to itself. For the same reason the pre-osculationals of the m th order for a stationary point include all those of lower order, while the stationary point has no post-osculation distinct from itself. By a proper pre-osculation of P of the m th order is meant a point Q whose m th post-osculation is the first to coincide with P .

The sixteen stationary points of τ form a closed group of points in the sense that the plane of any three of them meets τ further in one of them. This was proved in § 9.

We notice that if four points u_1, u_2, u_3 and u_4 of τ lie in a plane, then so do their m th post-osculationals. For, if

$$u_1 + u_2 + u_3 + u_4 \equiv 0 \pmod{\text{per.}},$$

then is $(-3)^m u_1 + (-3)^m u_2 + (-3)^m u_3 + (-3)^m u_4 \equiv 0 \pmod{\text{per.}}$.

Bearing this in mind, consider the 16×9^m points consisting of the sixteen stationary points and all of their pre-osculationals of the m th order (proper or otherwise). This group of points, then, contains the pre-osculationals of all the lower orders. Let the plane of any three points u_1, u_2, u_3 of the group meet τ further in u_4 . By taking at most m th post-osculationals, u_1, u_2 and u_3 give us three stationary points whose plane, as remarked above, must meet τ further in a stationary point u . Thus at most the m th post-osculation of u_4 is the stationary point u . Hence u_4 belongs to the group. Therefore we have the result that *the sixteen stationary points and all of their pre osculationals of the m th order (proper or otherwise) form a closed group of 16×9^m points.*

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ON THE PROJECTION OF THE GENERATORS OF A QUADRIC

By

ASGHAR HAMEED

(Communicated by Prof. C. V. H. Rao)

Introduction *

After proving that the projections of the generators of a quadric from a point on a plane envelop a conic (§1), I discuss the particular cases when this *envelope conic* may degenerate (§§2, 3). This is followed by a discussion on orthogonal projection (§6), and the behaviour of the envelope conic as the vertex of projection varies in space (§7). This envelope conic touches at n points, the projection of a curve of order n lying on the quadric (§8).

In particular, I obtain an ∞^1 of conics, each of which touches a plane quartic with two double points at four points (§9), and I show that a single conic is outpolar to each one of these ∞^1 of conics (§10). Also I prove that the quartic has eight bitangents (§9). Then I consider the cases when the vertex of projection takes some special positions (§§14-16).

Similarly, I get an ∞^2 of conics, each of which touches a plane cubic with one double point, at three points (§18), and discuss the number of these conics which pass through a point, or touch a line, etc. (§19). I also consider the behaviour of the envelope conics arising from the family of quadrics which pass through a given conic (§23), and from the families of quadrics which are reciprocals of the families already considered (§§17, 22, 24).

I wish to thank Professor C. V. Hanumanta Rao, whose guidance enabled me to write this paper.

1. Throughout this paper, we shall denote the vertex of projection by P , and the plane on which figures are projected by Π . Let Q be the given quadric, and g any one of its generators. The plane Pg meets Q in another generator s , and is the tangent plane to Q

* An abstract of this paper was read before the Second Subject Conference in Mathematics of the Punjab University on March 28, 1941.

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at the point common to g and s . Thus Pg is a tangent plane of the enveloping cone Σ of Q from P . Hence, *the projections of the generators of Q from P on Π envelop the section of Σ by Π* . This envelope conic (section of Σ by Π) will throughout be denoted by σ . It may be observed that a tangent line of σ is the projection of two generators of opposite systems of Q .

Let q be the section of Q by Π . Then we shall show that σ and q have double contact. For, if p be the section of Q by the polar plane of P w.r.t. Q , then the curve of intersection of Σ and Q consists of p taken twice. Thus q , which lies on Q , meets Σ in the two points, each taken twice, where q and p meet. Hence, q and σ touch at each of these two points.

2. Suppose P lies on Q , and g_1 and s_1 are the two generators of Q through P . Let g_1 and s_1 meet Π in G_1 and S_1 respectively. Since each g meets s_1 , it follows that the projection of each g passes through S_1 . Similarly the projection of each s passes through G_1 . Hence, *when P lies on Q , the envelope conic σ degenerates into the pair of points where the pair of generators of Q through P meet Π* . Also conversely, for if σ degenerates into two points G_1 and S_1 , then all the lines in Π , which pass through G_1 are the projections of the generators of Q which meet PG_1 , showing that PG_1 is a generator of Q ; and similarly that PS_1 is the other generator of Q through P .

3. If Q is a cone of vertex V , and V' is the projection of V from P on Π , then, obviously, the projections of all the generators of Q pass through V' . Conversely, if the projections of all the generators of a quadric Q pass through the same point V' , then as in §2, PV' is a generator of Q , belonging to both the families. This shows that the two families of generators of Q coincide, i.e., Q is a cone, and that V' is the projection of the vertex of Q . Hence, *σ degenerates into a pair of coincident points, if and only if, Q is a cone, and then σ consists of the projection of the vertex of Q taken twice*.

4. We have proved that σ and q have double contact (§1). In the particular case when Π is the polar plane of P w.r.t. Q , σ coincides with q . For, a quadric and its enveloping cone from a point having contact along the section of the quadric by the polar plane of the point, w.r.t. the quadric.

If σ degenerates into a pair of lines, then the enveloping cone of Q from P must consist of a pair of planes which will be coincident if P lies on Q , and distinct if Q is a cone. But in these two cases

we have seen that σ degenerates respectively into a pair of distinct points (§2), and a pair of coincident points (§3). Hence it follows that σ cannot degenerate into a pair of lines, whether distinct or coincident. In fact no envelope conic can degenerate into a pair of lines.

5. Let Π be taken parallel to the plane of p , the ring of contact of Σ and Q . Then it follows from §1, that the two points of contact of σ and q are at ∞ , namely the two points where q and p , whose planes are parallel, meet. Thus in this case, σ and q are similar and similarly situated conics. Now, all parallel plane sections of a quadric are similar and similarly situated; σ and p lie on Σ ; and q and p lie on Q . Hence each of σ and q is similar and similarly situated in regard to p whose plane is parallel to their common plane.

6. Let us now consider the case when the generators of Q are projected orthogonally on Π . Here, P is the point at ∞ on any line which is normal to Π . If P lies on Q , then the two generators of Q through P will both be perpendicular to Π . And conversely, if Π is perpendicular to a generator of Q , then P , being the point at ∞ on that generator, lies on Q . Thus it follows from §2, that the envelope conic σ degenerates into a pair of points, if and only if, Π is perpendicular to a generator of Q , in which case, of course, it will be perpendicular to another generator of Q , namely that of the other family which passes through the point at ∞ of the generator which is perpendicular to Π .

Since P lies at ∞ in a direction which is normal to Π , it follows that Π will coincide with the polar plane of P w.r.t. Q , if and only if, it is one of the principal planes of Q . Hence we see from §4 that, when projection is orthogonal, σ coincides with q , if and only if, Π is one of the principal planes of Q .* In particular, when Q is a sphere, σ and q coincide, if and only if, Π passes through the centre of Q .

7. From §1 it follows that when Π and Q are kept fixed and P is made to vary in space, the variable envelope conic σ envelopes the fixed conic q , always having double contact with it. There are ∞^3 conics which have double contact with q . These arise from the ∞^3 positions of P in space. There are an ∞^1 conics σ which touch q at two fixed points on it, and an ∞^2 conics σ which touch q at one fixed point on it. We proceed to enquire as to the behaviour of P corresponding to these families of the envelope conics σ .

* Bell, Co-ordinate Geometry of Three Dimensions, §104.

Let K_1 and K_2 be the points of contact of q and a particular envelope conic σ_1 which arises from a particular position P_1 of P . Then K_1 and K_2 lie on the ring of contact of Q and its enveloping cone Σ_1 from P_1 . Thus the tangent planes of Q at K_1 and K_2 pass through P_1 , which therefore lies on k , the line conjugate to $K_1 K_2$ w.r.t. Q . If P be taken anywhere on k , it is evident that the corresponding ring of contact will contain K_1 and K_2 . Hence it follows that the ∞^1 positions of σ , which touch q at any two fixed points on it, arise from the ∞^1 positions of P given by the line which is conjugate w.r.t. Q to the join of the two fixed points. In particular, this double contact may become a four-point contact when the two fixed points K_1 and K_2 coincide. In this case $K_1 K_2$ becomes a tangent line of Q lying in the plane Π , and so the corresponding locus of P becomes the line joining K_1 to the pole of Π w.r.t. Q .

Similarly, the ∞^2 envelope conics σ which have contact with q at a fixed point K on it, arise from the ∞^2 positions of P given by the tangent plane of Q at K . These ∞^2 conics σ envelop q . To get an ∞^1 conics σ which have contact with q at a fixed point K on it, and also envelop it, we have only to take P on any line lying in the tangent plane of Q at K , but not passing through the pole of Π w.r.t. Q .

8. Let C_n be a curve of order n lying on Q . Then it meets any plane section of Q in n points. Now the curve of intersection of Q and its enveloping cone Σ from P consists of the ring of contact p taken twice. Hence all the common points of Σ and C_n , which lie on Q , consist of the n points, each taken twice, where C_n meets p . Thus C_n touches Σ at these n points. If C_n' be the projection of C_n from P on Π , it follows that the envelope conic σ touches C_n' at n points which are the projections of the n points of contact of Σ and C_n . Thus the $2n$ points of intersection of σ and C_n' are accounted for.

9. Consider now the twisted quartic curve τ which is common to a pencil (∞^1) of quadrics. It is known that a generator of any one of these quadrics is a bisecant of τ , and that one of these quadrics passes through a point of general position in space. Hence considering the quadric of the pencil which passes through P , and its two generators thereat, we see that two bisecants can be drawn to τ from P . Thus the projection of τ from P on Π is a quartic curve τ' having two double points. Conversely, any plane quartic curve

with two double points, is the projection of a τ .* Thus, from the previous article it follows that, corresponding to the ∞^1 of quadrics which pass through τ , we get an ∞^1 of envelope conics $\dagger$ σ each of which touches the general plane quartic with two double points, at four points. The two double points of τ' constitute the degenerate σ corresponding to that quadric of the pencil which passes through P.

Now, let V_1, V_2, V_3 and V_4 be the vertices of the four cones which belong to the pencil of quadrics determined by τ . Let their projections from P on Π be respectively V'_1, V'_2, V'_3 and V'_4 . Each of these four points, taken twice, is the degenerate σ corresponding to that cone of the pencil, whose vertex projects into it (§3), and from each of these four points, a pair of bitangents can be drawn to τ' . For, a pair of tangent planes can be drawn from P to any one of the four cones of the given pencil of quadrics. Each of these tangent planes touches the cone along a generator which is a bisecant of τ , and hence it meets the plane Π in a bitangent of τ' . Also conversely, let l be a bitangent of τ' , touching it at K'_1 and K'_2 , which are the projections of the points, say K_1 and K_2 , of τ . Then the plane Pl touches τ at each of K_1 and K_2 . Thus the section by the plane Pl of the quadric which passes through τ and its bisecant K_1K_2 consists of K_1K_2 taken twice. The quadric must therefore be a cone, and Pl , a tangent plane thereof. The bitangent l , then, belongs to some one of the four pairs already determined. Thus the general plane quartic curve with two double points has eight bitangents. Also, since ${}^3C_2=28$, we see that there are in all twenty-eight points in the plane Π , from each of which, a pair of bitangents can be drawn to τ' .

10. We shall now show that the line joining the two double points of τ' meets the ∞^1 envelope conics σ , found in the previous article, in pairs of points belonging to an involution. For, let the two bisecants of τ from P meet it in K_1, K_2 and L_1, L_2 respectively. Let P_k be the harmonic conjugate of P w.r.t. K_1 and K_2 , and P_l , the harmonic conjugate w.r.t. L_1 and L_2 . Then the polar planes of P w.r.t. the quadrics of the pencil determined by τ , pass through the line P_kP_l . Hence the pair of points, in which P_kP_l meets any quadric of the pencil, also lie on the enveloping cone of the quadric from P. But the pairs of points in which a line meets the quadrics

* Baker, Principles of Geometry, Vol. IV, p. 89.

† Similar families of conics are known to exist for the general plane quartic curve, and are derived from a cubic surface (Baker, *ibid.*, Vol. III, p. 199).

of a pencil belong to an involution. Therefore the pairs of points, in which $P_k P_l$ meets the enveloping cones from P of the quadrics which pass through τ , belong to an involution. Hence projecting on the plane Π we get the result stated at the beginning of this article. It may be observed that *the pair of double points of τ' also belongs to the stated involution*. For, the two double points are the projections of P_k and P_l which are the points in which the line $P_k P_l$ meets that quadric of the pencil which passes through P . Similarly, the pairs of points, in which the join of the double points of τ' meets the pairs of bitangents of τ' from V'_1, V'_2, V'_3 and V'_4 , also belong to the stated involution. The united elements of this involution will be denoted by V'_5 and V'_6 . We shall show (§11) that $V'_1, \dots, V'_6$ lie on a conic, which is outpolar to each one of the ∞^1 envelope conics σ .

11. Let V_5 and V_6 be the united elements of the involution determined by the pairs of points in which the line $P_k P_l$ meets the quadrics of the pencil fixed by τ . Then evidently V'_5 and V'_6 of the previous article are the projections of V_5 and V_6 .

Now, it is easy to show that the locus of the poles of an arbitrary plane w.r.t. the quadrics of a pencil is a cubic curve.* The three points where that plane meets the cubic curve are the points of contact of that plane with the three quadrics of the pencil which are touched by it. Thus the cubic curve, which is the locus of the poles of the plane $P P_k P_l$ w.r.t. the quadrics of the given pencil, meets the plane in the points P, V_5 and V_6 . Now the lines conjugate to $P_k P_l$ (or $V_5 V_6$) w.r.t. the quadrics of the pencil, all pass through P . Hence they generate the cone whose vertex is P , and which passes through the cubic curve found as the locus of the poles of $PP_k P_l$. This cone passes through V_5 and V_6 , because, as already remarked, these two points lie on the cubic curve. Also it passes through V_1, V_2, V_3 and V_4 , for the line conjugate to a given line w.r.t. a cone must pass through the vertex of the cone. Thus, by taking section of this cone by the plane Π , we see that $V'_1, \dots, V'_6$ lie on a conic.

Taking Q as the general quadric of the pencil fixed by τ , let p be the ring of contact of Q and its enveloping cone Σ from P . Then evidently the pole of the $P_k P_l$, which lies in the plane of p , w.r.t. the conic p must lie on the line conjugate to $P_k P_l$ w.r.t. Q . Thus the pole of $P_k P_l$ w.r.t. p lies on the cone generated by the lines

* For the reciprocal statement, see Baker, *ibid.*, Vol. III, p. 103.

conjugate to $P_k P_l$ w.r.t. the quadrics of the pencil. Hence, since p projects into the σ corresponding to Q , it follows that the conic determined by $V_1', \dots, V_6'$ is the locus of the poles of the join of the double points of τ' , w.r.t. the ∞^1 of envelope conics σ . Now V_5' and V_6' are conjugate w.r.t. each of the ∞^1 conics σ . Hence, if V' be the pole of $V_5' V_6'$ w.r.t. any one of the ∞^1 conics σ , the $\Delta V' V_5' V_6'$ is inscribed in the conic $V_1' V_2' V_3' V_4' V_5' V_6'$ and is self-conjugate w.r.t. that particular σ . This proves the statement made at the end of the previous article, completely.

12. We now prove the following results in regard to the ∞^1 of envelope conics σ each of which touches τ' at four points:—

(i) *One and only one of the conics σ touches τ' at a given point on it.* Let K' be any point of τ' , and let it be the projection of the point K of τ . Denote the tangent line of τ' at K' by k' . Then the plane Pk' touches τ at K and thus meets it at two other points. A σ which touches τ' at K' must arise from a quadric of the pencil which touches the plane Pk' at K . There is one and only one quadric with this property, namely, the quadric whose generators through K are the joins of K with the other two points in which Pk' meets τ . Hence the required result.

This could also be proved by saying that a σ which touches τ' at K' must arise from a quadric, w.r.t. which the polar plane of P passes through K . And there is one and only one quadric of the pencil having this property.

(ii) *Two and only two of the conics σ pass through an arbitrary point in the plane II.* For, if P' be any point in the plane II, then a σ which passes through P' must arise from a quadric of the pencil, whose enveloping cone from P has PP' as a generator. Thus PP' must be a tangent line of the quadric. But an arbitrary line touches two and only two quadrics of a pencil. Hence, the two quadrics of the pencil, which touch PP' , give rise to two envelope conics σ which pass through P' . This proves the required result.

(iii) *Three and only three of the conics σ touch an arbitrary line in the plane II.* For, let l' be any line in the plane II. Then a σ , which touches l' , must arise from a quadric of the pencil, of which Pl' is a tangent plane. But an arbitrary plane touches three and only three quadrics of the pencil, which correspond to the three pairs of lines that can be drawn through the four points where the plane meets τ . These three quadrics, and these alone, give rise to the conics σ which touch l' . And this was to be proved.

It may be observed that in §10 we proved that the line joining the two double points of τ' meets the conics σ in pairs of points belonging to an involution. Thus, corresponding to the two united elements of this involution, there are two conics σ which touch that line. The third conic σ touching that line is the degenerate σ consisting of the two double points of τ' .

13. The quartic curve τ , considered above, may in some cases degenerate into simpler curves, *e.g.*, it may consist of a cubic γ and a line l which, of course, must be a bisecant of γ . If γ' and l' be the projections of γ and l respectively, from P on Π , then each of the ∞^1 envelope conics σ touches the plane cubic γ' at three points, and the line l' at one point. Similarly, we may consider other degenerate cases of τ . In particular, it may consist of four lines, and then the ∞^1 envelope conics σ belong to a range of conics, determined by the projections of the four lines constituting τ , and have a common self-conjugate triangle.

14. In this article we shall consider the cases when P , the vertex of projection, takes some special positions in regard to the quartic curve τ .

(a) Let P be taken on a face of the tetrahedron $V_1V_2V_3V_4$ formed by the vertices of the four cones which belong to the pencil of quadrics determined by τ . Say, P is taken on the face $V_2V_3V_4$. Then since $V_1V_2V_3V_4$ is a self-conjugate tetrahedron for all the quadrics of the pencil, it follows that the line P_kP_l (§10) passes through V_1 . Hence, the lines conjugate to P_kP_l w.r.t. the quadrics of the pencil, lie in the plane $V_2V_3V_4$ which is the polar plane of V_1 w.r.t. each quadric of the pencil. Thus in this case, *the locus of the poles of the line joining the double points of τ' , w.r.t. the ∞^1 envelope conics σ is the line $V_2'V_3'V_4'$* . And the conic $V_1'V_2'V_3'V_4'/V_5'V_6'$ of §10 breaks up into a pair of lines $V_2'V_3'V_4'$ and $V_5'V_6'$. It may be remarked that, in this case, one of the points V_5 and V_6 of §11 coincides with V_1 and the other with the point where the line P_kP_l meets the plane $V_2V_3V_4$. Hence one of the points V_5' and V_6' is the point V_1' and the other is the point where the line $V_2'V_3'V_4'$ meets the line joining the double points of τ' .

(b) Let P be taken at a vertex of the tetrahedron $V_1V_2V_3V_4$. Then the ∞^1 of envelope conics σ belong to a pencil of conics. For taking P at V_1 say, we know that it has the same polar plane w.r.t. each quadric of the pencil, namely, the plane $V_2V_3V_4$. Let $V_2V_3V_4$ meet τ in the points X_1, X_2, X_3 and X_4 .

Then the conics, in which the plane $V_2V_3V_4$ meets the quadrics of the pencil, belong to the pencil of conics determined by the points $X_1, \dots, X_4$. But these conics project into the envelope conics σ . Hence if $X_1', \dots, X_4'$ be the projections from P on Π of the points $X_1, \dots, X_4$ respectively, it follows that the conics σ belong to the pencil determined by $X_1', \dots, X_4'$, which was to be proved. Also τ projects into a conic (taken twice), which is the section by Π of the cone of the pencil whose vertex is P . This conic evidently belongs to the pencil of conics σ .

(c) If τ degenerates into a cubic curve and a line meeting the cubic in P_1 and P_2 , then there are only two cones which belong to the pencil of quadrics, their vertices being P_1 and P_2 respectively. If P is taken at any one of these two points, the envelope conic σ , corresponding to any quadric of the pencil, degenerates into a pair of points (§2), of which one point remains fixed being the point of intersection of Π and P_1P_2 , and the other moves on the conic into which the cubic curve projects.

(d) If τ degenerates into two conics meeting in K_1 and K_2 , then all the quadrics of the pencil have the same tangent planes at K_1 and K_2 . In this case also there are only two cones belonging to the pencil of quadrics, their vertices P_1 and P_2 being the united elements of the involution fixed by the pairs of points in which the quadrics of the pencil are met by the line of intersection of the common tangent planes at K_1 and K_2 . If P is taken at any one of the two points P_1 and P_2 , the envelope conics σ have contact with one another at each of the two fixed points where Π meets PK_1 and PK_2 . It may be observed that at these two points the conics σ also touch the section by Π of the cone of the pencil, whose vertex is P . The common pole w.r.t. the conics σ of the line, joining these two points of contact, is the point where P_1P_2 meets Π , and therefore remains fixed whether P is taken at P_1 or P_2 . These results hold when P is taken anywhere on the line P_1P_2 .

The cases when one or both of the two conics, into which τ degenerates, consist of pairs of lines may be discussed similarly.

15. Let us now consider what happens when P is taken anywhere on the quartic curve τ . Through P there pass two generators of each quadric of the pencil, and these generators are bisecants of τ . Also, a bisecant of τ is a generator of one and only one quadric of the pencil. If any plane be drawn through P , it will meet τ in three other points which, joined to P , give three bisecants of τ passing



through P and lying in that plane. Thus the pairs of generators through P of the quadrics of the pencil lie on a cubic cone. Also, *these pairs of generators determine an involution on the cubic cone*, for a quadric is uniquely fixed by the quartic curve τ and a bisecant thereof. The planes of the pairs of generators, being tangent planes at P to the quadrics of the pencil, have in common the tangent line at P to τ , which line is also a generator of the cubic cone found above.

Now, since in this case P lies on each quadric of the given pencil, it follows (§ 2) that σ for each quadric degenerates into a pair of points where the pair of generators through P of that quadric meet Π . *These pairs of points into which the conics σ degenerate, determine an involution on the elliptic cubic curve of intersection of Π and the cubic cone found above*, the line joining the two points of each pair passing through the fixed point A' of the cubic curve where Π meets the tangent line at P to τ . This follows from the previous paragraph.

The united elements of the involution, determined by the degenerate conics σ , will be points of the cubic curve, tangents at which to the curve pass through A' . These united elements will be the degenerate conics σ corresponding to the cones belonging to the given pencil of quadrics (§ 3). Thus we have established *the equivalence of the two statements* that four cones belong to a pencil of quadrics, and that four tangent lines can be drawn to a plane elliptic cubic from a point on it, apart from the tangent line at that point.

16. In continuation of the last article, we now consider the following particular cases when τ degenerates into simpler curves:—

(a) Let τ consist of a twisted cubic curve and a line, which of course will be a bisecant of the cubic curve. Let P be taken anywhere on the twisted cubic. Then each envelope conic σ degenerates into a pair of points (§ 2), of which one lies on the conic into which the cubic curve projects and the other lies on the line in which the bisecant of the cubic curve projects. These line and conic meet in two points each of which taken twice is a degenerate σ . This is equivalent to the fact that only two cones belong to the pencil of quadrics in this case.

(b) Let τ consist of two conics meeting in K_1 and K_2 , and let P be taken on any one of the two conics. The other conic meets each of the two generators through P of any quadric of the pencil. Hence, the pairs of points, in which the envelope conics σ degenerate, deter-

mine an involution on the conic into which the second of the two given conics projects from P on Π . Such an involution has two united elements, which is thus equivalent to the fact that in the present case only two cones belong to the pencil of quadrics. The centre of the involution is the point where Π meets the tangent line at P to the conic on which P is taken.

If however, P be taken at K_1 or K_2 , the involution works on the line of intersection of Π and the common tangent plane of the quadrics of the pencil, at P .

(c) When τ consists of a conic and a pair of intersecting lines, each of which will meet the conic in a point, let us take P on any one of the two lines. Then, of the two points into which a σ degenerates, one remains fixed, being the point of intersection of Π and the line on which P is taken, and the other moves on the conic into which the given conic projects from P on Π . Evidently the two points, which constitute a σ , coincide only once, namely, at the fixed point just determined. This is so, because in the present case, only one cone belongs to the pencil of quadrics.

If P be taken on the given conic, then of the two points into which a σ degenerates, one will lie on one fixed line and the other on another fixed line. These two fixed lines are the projections of the two given lines which form a part of τ . Again we see that the two points constituting a σ coincide just once, namely at the point where the two fixed lines intersect.

(d) When τ consists of four lines forming a skew quadrilateral, and P is taken on any one of them, then the edge opposite to this line will be met by the other generator through P of each quadric of the given pencil. Thus, of the two points into which a σ degenerates, one is fixed, being the point of intersection of Π and the line on which P is taken, and the other moves on the projection of the opposite edge which obviously does not pass through the fixed point. It follows then, that the two points which constitute a σ cannot coincide. This is equivalent to the fact, that in the present case, no cone belongs to the given pencil.

If P be taken on any one of the four vertices of the skew quadrilateral, then all the quadrics of the pencil have the same two generators through P . Thus all the conics σ degenerate into the same pair of points, where the common generators through P meet Π .

17. We shall now consider a range of quadrics obtained as the reciprocals of the quadrics of a pencil. Since an arbitrary plane

meets the quadrics of a pencil in four common points, it follows that, from an arbitrary point P , four planes can be drawn to touch all the quadrics of the range. These four planes are thus the common tangent planes of the enveloping cones from P to the quadrics of the range. Hence the envelope conics σ , arising from the quadrics of the range, belong to a range of conics determined by the four lines where Π meets the four common tangent planes through P .

Now an arbitrary plane touches three quadrics of a pencil. The three pairs of generators of these quadrics, lying in the plane, are the three pairs of lines which pass through the four points in which that plane meets all the quadrics of the pencil. Hence, reciprocally, we see that through the arbitrary point P , three quadrics of the range pass. And that the pairs of generators of these quadrics, through P are given by the three pairs of lines which lie in the four common tangent planes through P of the quadrics of the range. Hence the degenerate conics σ , corresponding to the three quadrics of the range which pass through P , are the three pairs of opposite vertices of the quadrilateral formed by the four lines which determine the range of conics σ . •

In particular the remarks of this article apply to a family of confocal quadrics which is a particular type of a range of quadrics.

18. Let γ be a twisted cubic curve. Then through the arbitrary point P a unique bisecant can be drawn to γ . Thus the projection of γ on the plane Π is a cubic curve γ' which has one double point. Now an ∞^2 quadrics can be drawn to pass through γ . Hence the ∞^2 envelope conics σ , arising from these ∞^2 quadrics, are a family of conics each of which touches γ' at three points (§ 8). We shall now show that these are the only conics which touch γ' at three points.

For, let a conic σ_1 touch γ' at the points K_1' , K_2' and K_3' . Let these three points be the projections respectively of K_1 , K_2 and K_3 which lie on γ . Let p_1 be the conic section by the plane $K_1K_2K_3$ of the cone whose vertex is P and whose base is σ_1 . Then p_1 meets γ in K_1 , K_2 and K_3 . Therefore taking two other points of p_1 it is clear that a unique quadric Q can be drawn to pass through γ and p_1 . The tangent plane of Q at K_1 must touch γ and p_1 , and is thus the tangent plane at K_1 to the cone whose vertex is P and whose base is σ_1 . Hence the tangent plane to Q at K_1 passes through P . So also for the tangent planes at K_2 and K_3 . Hence P is the pole of the plane $K_1K_2K_3$ w.r.t. Q , showing that the cone whose vertex is P

and whose base is p_1 (or σ_1) is the enveloping cone of Q from P . Thus σ_1 is the envelope conic σ corresponding to the quadric Q which passes through γ , i.e., σ_1 is one of the ∞^2 conics found above. This proves the required result.

19. We now state and prove the following results in regard to γ' , the plane cubic (rational) with one double point, and the ∞^2 conics σ found in the previous article:—

(a) *An ∞^1 of the conics σ touch γ' at a given point on it.* For, let K_1' be any point of γ' , and let it be the projection of the point K_1 of γ . Then the conics σ which touch γ' at K_1' must arise from the quadrics which pass through γ and w.r.t. each of which the polar plane of P passes through K_1 . We have then to show that there are an ∞^1 of such quadrics. Now, the polar planes of P w.r.t. the quadrics, which contain γ , pass through a fixed point P_1 which is the fourth harmonic of P w.r.t. the two points where the unique bisecant of γ through P meets γ . Thus the polar planes of P w.r.t. the quadrics whose conics σ touch γ' at K_1' must pass through the line $P_1 K_1$. There are an ∞^1 of planes which pass through $P_1 K_1$, thus giving the ∞^1 of the required quadrics. This proves the desired result.

(b) *There are two conics σ , each of which touches γ' at a given point on it and has four-point contact with it at some other point.* For, such a σ will arise from a quadric of the (∞^2) family fixed by γ , the polar plane of which w.r.t. P , not only passes through $P_1 K_1$ (see the previous case) but also touches γ at a point other than K_1 . Such a plane is obviously a tangent plane from P_1 to the cone which passes through γ and has K_1 as its vertex. Since only two tangent planes can be drawn from P_1 to the cone, we get two quadrics (corresponding to these two planes) which give rise to two conics σ having the stated property. And this was to be proved.

The two tangent planes coincide if, and only if, P_1 lies on the cone which passes through γ and whose vertex is K_1 . Then $P_1 K_1$ will be the bisecant of γ from P_1 . Thus $P_1 K_1$, in this case, coincides with PP_1 showing that K_1' is the double point of γ' . Also K_1 has now two positions, say K_{11} and K_{12} , which are the points where the bisecant from P to γ meets it. The tangent plane at P_1 to the cone, which passes through γ and has K_{11} as vertex, is the polar plane of P w.r.t. this very cone, for $P_1 K_{11}$ passes through P . And the σ corresponding to this cone is the double point of γ' , taken twice. Similarly also for the cone which passes through γ and has K_{12} as vertex. Thus

when K_1' is the double point of γ' , each of the two conics σ discussed above consists of the double point taken twice. Apart from this, *the two conics σ found above will always be distinct.*

(c) *One and only one σ touches γ' at each of its two given points.* For, let K_1' and K_2' be the two given points of γ' . Let them be projections respectively of the points K_1 and K_2 which lie on γ . Then by the argument used in case (a), the quadric, which passes through γ and w.r.t. which $P_1K_1K_2$ is the polar plane of P , is the only quadric which gives rise to a σ touching γ' at K_1' and K_2' . This proves the required result.

(d) *Among the ∞^2 conics σ , there are three families, each of ∞^1 conics, such that the conics of these families touch any arbitrary line in the plane II.* For, let l' be an arbitrary line in the plane II, meeting γ' in K_1' , K_2' and K_3' . Let these three points be respectively the projections of the points K_1 , K_2 and K_3 which lie on γ . Then $K_1K_2K_3$ is a triangle. Take any side of this triangle, say K_2K_3 . Then γ and its bisecant K_2K_3 constitute a degenerate quartic curve, and (§ 13) we get an ∞^1 of the conics σ each of which touches γ' at three points and l' at one point. Thus corresponding to the three sides of the triangle $K_1K_2K_3$ we get the three families stated above.

(e) *Three of the conics σ touch an arbitrary line in the plane II, at a given point on it.* For, in the notation of the previous case, let K' be the given point on l' , and let PK' meet K_2K_3 in K . Then the unique quadric, which passes through γ and K_2K_3 , and w.r.t. which the polar plane of P passes through K , gives rise to a σ touching l' at K' . Thus corresponding to the three families of the previous case, we get the three conics σ stated above.

(f) *Nine of the conics σ touch two arbitrary lines in the plane II.* For, in continuation of case (d), let l_1' be another arbitrary line in the plane II, meeting γ' in K'_{11} , K'_{12} and K'_{13} . Let these points be the projections respectively of K_{11} , K_{12} and K_{13} which lie on γ . Then a unique quadric can be drawn to pass through γ and its two bisecants K_2K_3 and $K_{12}K_{13}$. The σ given by this quadric touches both l' and l_1' . Corresponding to the nine ways of grouping one side of the triangle $K_1K_2K_3$ with one side of the triangle $K_{11}K_{12}K_{13}$, we get the nine conics σ which touch l' and l_1' , which was to be proved.

(g) *Three of the conics σ touch γ' at a given point on it and also touch an arbitrary line in the plane II.* This may be proved as case (e).

(h) An ∞^1 of the conics σ pass through an arbitrary point in the plane Π , and in threes they have the same tangent line at that point. Since an ∞^1 of lines can be drawn to pass through a point and to lie in a plane containing the point, the result stated above follows from case (e).

(i) Two and only two conics σ touch γ' at a given point on it and pass through a given point in the plane Π . For, the conics σ , which touch γ' at a fixed point on it, arise from a certain pencil of quadrics [this follows from the argument used in case (a)]. Two and only two quadrics of this pencil touch the line joining P to the given point in the plane Π . The conics σ corresponding to these two quadrics have the stated property. This proves the required result.

(j) Four of the conics σ pass through two arbitrary points in the plane Π . To prove this it is enough to show that the two lines, which join the two arbitrary points in the plane Π to the point P , touch four of the quadrics which pass through γ . Hence we shall prove the following:—

Of the ∞^2 quadrics which pass through γ there are just four which touch each of two given lines in space. For, let the lines be AB and CD . If K be any point of AB , then seven arbitrary points of γ and K , taken twice (on AB), give us nine points which determine uniquely the quadric* which passes through γ and touches AB at K . Thus of the quadrics which pass through γ , an ∞^1 touch AB . Let any one of these ∞^1 quadrics meet CD in L_1 and L_2 . Then we shall show that there is a (2, 2) correspondence between L_1 and L_2 , i.e., if L be any point on CD , then two of the ∞^1 quadrics pass through L . For, any such quadric is touched by AB , and belongs to the pencil of quadrics which pass through the quartic curve consisting of γ and its unique bisecant through L , and two quadrics of a pencil touch an arbitrary line. Thus it is proved that there is a (2, 2) correspondence between L_1 and L_2 . Such a correspondence has four united elements, showing that four of the ∞^1 quadrics touch CD , which was to be proved.

20. There are an ∞^1 of quadrics which pass through γ and P , being the quadrics of the pencil which is determined by the quartic

* This quadric may also be located by means of its two generators through K . One of them is the unique bisecant of γ through K , and the other is the line joining K to the third point where γ is met by the plane passing through AB and the unique bisecant of γ through K , since this plane is the tangent plane at K to the required quadric.

curve consisting of γ and its unique bisecant through P. The envelope conics σ corresponding to these ∞^1 of quadrics degenerate each into a pair of points (§ 2), of which one point is fixed, namely, the double point of γ' and the other moves on γ' .

We know that if γ lies on a cone, it must pass through the vertex of the cone, and conversely, a quadric cone can be drawn to pass through γ with any one of its points as vertex. The envelope conics σ , corresponding to the ∞^1 of cones which pass through γ , break up into pairs of coincident points (§ 3) which are the points of γ' . From this we can prove that *from an arbitrary point K' of γ' we can draw two tangents to γ' , apart from the tangent at K'* . For, let K be the corresponding point of γ . Then the plane passing through PK' and a tangent line of γ' from K' must be a tangent plane from P to the cone which passes through γ and whose vertex is K. Since only two such tangent planes can be drawn, we get only two lines which are tangents to γ' from K' , and this was to be proved. [This, of course, can be proved otherwise also. For, the polar conic of K' w.r.t. γ' touches γ' at K' and passes through its double point. Thus it meets γ' in only two other points. And tangents to γ' at these two points pass through K' .]

21. We shall now show that *γ' has three points of inflexion and that these three points lie on a line*. For, three osculating planes of γ pass through the arbitrary point P. These planes meet Π in three lines each of which meets γ' in three coincident points, giving us a point of inflexion of γ' . Also conversely, the plane joining P to a line of inflexion of γ' must be an osculating plane of γ . Hence there are only three points of inflexion of γ' . That they are collinear follows from the fact* that the plane of the three points, at which the osculating planes of γ pass through P, also passes through P.

Now, the osculating plane of γ , at a point K on it, is a tangent plane of the cone which passes through γ and has K as its vertex. Thus if K' be a point of inflexion of γ' , then one of the two tangents from K' to γ' , found in the previous article, coincides with the line of inflexion at K' . It follows, then, that from a point of inflexion of γ' , only one tangent can be drawn to γ' , apart from the line of inflexion at that point. [Otherwise thus: The polar conic of K' w.r.t. γ' consists of two lines, one of which is the line of inflexion at K' . The other passes through the double point of γ' and thus meets it in only one other point. The tangent to γ' at this point passes through K' .]

* Baker, *ibid.*, Vol. III, Ex 2, p. 134.

22. Consider now the family (∞^2) of quadrics obtained as the reciprocal of the family of quadrics which pass through a twisted cubic γ . Since an arbitrary plane meets γ in three points it follows that three common tangent planes can be drawn from the arbitrarily chosen point P to the quadrics of the family under consideration. Hence *the envelope conics σ arising from the quadrics of this family are the conics which touch three fixed lines*, which are the sections by the plane Π of the three common tangent planes through P .

23. We now come to the family (∞^4) of quadrics which have a conic C in common. If we take the sections by the plane of C , of the enveloping cones from P of the ∞^4 of quadrics, then it follows that each of these conic sections has double contact with C (§1). Hence, if C' be the projection of C from P on Π , we see that *the envelope conics σ , arising from the ∞^4 of quadrics, have double contact with the fixed conic C'* .

The number of conics having double contact with a given conic is ∞^3 , while the number of conics σ which have double contact with C' is ∞^4 . This can be accounted for by the fact that all the ∞^4 conics σ are not distinct, an ∞^1 of them being coincident with each one of them. For, *an enveloping cone of one of the ∞^4 quadrics is an enveloping cone of an ∞^1 of them*. This may be proved as follows:—

Let Σ be an enveloping cone of one of the given ∞^4 of quadrics. Then it must touch C at two points, say, K_1 and K_2 . Hence the planes of the rings of contact of Σ and those of the ∞^4 quadrics, which have Σ as an enveloping cone, must pass through K_1K_2 . An ∞^1 of planes can be drawn through K_1K_2 . We have to show that a single quadric can be drawn which passes through C and has contact with Σ along the conic k where Σ is cut by an arbitrary plane through K_1K_2 . Let R be any point on C . Then we can draw a single quadric having ring contact with Σ along k , and passing through R . The section of this quadric by the plane of C must touch Σ at K_1 and K_2 because these are the two points where this plane meets the ring of contact k . Hence this section touches C at K_1 and K_2 . And since it meets C further in the point R , it coincides with C . Thus the quadric we have constructed passes through C and is the only quadric of the given family, which has ring contact with Σ along k . Corresponding then, to the ∞^1 planes through K_1K_2 , we get the ∞^1 quadrics from the given family which have Σ as an enveloping cone, which was to be proved.

24. Let us now consider the family of quadrics which is the reciprocal of the family of the previous article. The present family, then, consists of the ∞^4 quadrics which have a common fixed enveloping cone S . Two tangent planes of S pass through the arbitrary point P . These are thus the common tangent planes from P of all the quadrics of the family under consideration. It follows, then, that *the envelope conics σ , arising from the quadrics of the present family, touch two fixed lines*, which are the lines where Π meets the two common tangent planes through P . The number of conics which touch each of two fixed lines is ∞^3 , while the number of conics σ is ∞^4 . This, as before, can be accounted for by the fact that all the conics σ are not distinct, an ∞^1 of them being coincident with each one of them. For, *an enveloping cone of one of the ∞^4 quadrics of the present family is an enveloping cone of an ∞^1 of them*. This may be proved as follows:—

Let Σ be an enveloping cone of any one of the ∞^4 of quadrics. Then the two tangent planes of S from the vertex of Σ must also be tangent planes of Σ . Thus Σ and S have two common tangent planes. We have only to show that both of them are enveloping cones of an ∞^1 of quadrics. This is at once seen to be the case, by reciprocating the statement that an ∞^1 of quadrics can be drawn to pass through two conics which intersect in two points. Hence the required result is proved.

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CORRECTION

In the paper ' On a Diophantine Equation ' by N. M. Basu, in the
Bull. Cal. Math. Soc., Vol. XXXII (1940), pp. 16-20

<i>Page</i>	<i>Line</i>	<i>Read</i>	<i>For</i>
16	3	$A_k X_k$	A_{k+1}
16	15	get	set
17	2	$x_1^{l_1}, x_2^{l_2}, \dots, x_{k+1}^{l_{k+1}}$	$x_1^{l_1} + x_2^{l_2} + \dots + x_{k+1}^{l_{k+1}}$
17	26	Y_1^g	Y^g
20	16	Y_1^g	Y_1^g



ON AN INTERPOLATION FORMULA CONNECTED WITH A DEFINITE INTEGRAL IN N -VARIABLES

BY

P. N. DAS GUPTA

Introduction

Burnside* had indicated that

$$\begin{aligned} & \int_{-1}^1 \int_{-1}^1 f(x, y) dx dy \\ &= \frac{4}{49} \left\{ f\left(\sqrt{\frac{7}{9}}, \sqrt{\frac{7}{9}}\right) + f\left(\sqrt{\frac{7}{9}}, -\sqrt{\frac{7}{9}}\right) \right. \\ & \quad + f\left(-\sqrt{\frac{7}{9}}, \sqrt{\frac{7}{9}}\right) + f\left(-\sqrt{\frac{7}{9}}, -\sqrt{\frac{7}{9}}\right) \left. \right\} + \frac{40}{49} \left\{ f\left(\sqrt{\frac{7}{15}}, 0\right) \right. \\ & \quad + f\left(-\sqrt{\frac{7}{15}}, 0\right) + f\left(0, \sqrt{\frac{7}{15}}\right) + f\left(0, -\sqrt{\frac{7}{15}}\right) \left. \right\}, \end{aligned}$$

where $f(x, y)$ is an integral polynomial of degree ≤ 5 .

The object of the present paper is to extend the result to the case of an integral polynomial in n -variables of degree $\leq m$, an odd integer.

1. Let $f(x_1, x_2, \dots, x_n) = \sum a_{rst\dots} x_1^r x_2^s x_3^t \dots$

where $r + s + t + \dots = 5, 4, 3, 2, 1, 0$.

So, on the assumption that degree of $f \leq 5$,

$$\begin{aligned} & \int_{-1}^1 \int_{-1}^1 \dots \int_{-1}^1 f(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n \\ &= \frac{2^n}{5} \left(a_{4000\dots} + a_{0400\dots} + \dots \right) + \frac{2^n}{9} \left(a_{2200\dots} + a_{0202\dots} + \dots \right) \\ & \quad + \frac{2^n}{3} \left(a_{2000\dots} + a_{020\dots} + a_{002\dots} + \dots \right) + 2^n a_{000\dots} \\ &= 2^n \left(\frac{\alpha}{5} + \frac{\beta}{9} + \frac{\gamma}{3} + \delta \right) \dots \quad (1) \end{aligned}$$

where $\alpha = a_{4000\dots} + a_{0400\dots} + \dots$; $\beta = a_{2200\dots} + a_{0202\dots} + \dots$;
 $\gamma = a_{2000\dots} + a_{020\dots} + a_{002\dots} + \dots$; $\delta = a_{000\dots}$.

* W. Burnside, *Mess. of Math.*, (2), 37 (1908).

Now suppose

$g(a) = f(a, a, \dots, a) + f(-a, a, \dots, a) + \dots = \sum f(\pm a, \pm a, \pm a, \pm a, \dots, \pm a)$,
a summation of 2^n terms.

$$\text{So,} \quad g(a) = 2^n (a^4 \alpha + a^4 \beta + a^2 \gamma + \delta). \quad \dots (2)$$

And suppose, ..

$$h(b) = f(b, 0, 0, 0, \dots) + f(-b, 0, 0, 0, \dots) + f(0, b, 0, 0, \dots) \\ + f(0, -b, 0, 0, \dots) + \dots$$

a summation of $2n$ terms,

$$\text{so that} \quad h(b) = 2(b^4 \alpha + b^2 \gamma + n\delta). \quad \dots (3)$$

If now we assume

$$\lambda g(a) + \mu h(b) = \frac{2^n \alpha}{5} + \frac{2^n \beta}{9} + \frac{2^n \gamma}{3} + 2^n \delta, \quad \dots (4)$$

we get the following equations for determining λ, μ , viz.,

$$\left. \begin{aligned} 2^n a^4 \lambda + 2\mu b^4 &= \frac{2^n}{5}, \\ 2^n a^4 \lambda &= \frac{2^n}{9}, \\ \lambda \cdot 2^n a^2 + \mu \cdot 2b^2 &= \frac{2^n}{3}, \\ \lambda \cdot 2^n + \mu \cdot 2n &= 2^n. \end{aligned} \right\} \quad \dots (5)$$

$$\text{Solving,} \quad b^2 = \frac{4+5n}{30}; \quad a^2 = \frac{5n+4}{15n-12}; \quad \dots$$

λ, μ have corresponding values.

Hence from (1) and (4),

$$\int_{-1}^1 \int_{-1}^1 \dots \int_{-1}^1 f(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n \\ = \frac{(15n-12)^2}{9(5n+4)^2} \left\{ f\left(\sqrt{\frac{5n+4}{15n-12}}, \sqrt{\frac{5n+4}{15n-12}}, \dots\right) \right. \\ \left. + f\left(-\sqrt{\frac{5n+4}{15n-12}}, \sqrt{\frac{5n+4}{15n-12}}, \dots\right) + \dots \text{ to } 2^n \text{ terms} \right\} \\ + \frac{20 \times 2^{n+1}}{(4+5n)^2} \left\{ f\left(\sqrt{\frac{4+5n}{30}}, 0, 0, 0, \dots\right) \right. \\ \left. + f\left(-\sqrt{\frac{4+5n}{30}}, 0, 0, 0, \dots\right) + \dots \text{ to } 2n \text{ terms} \right\}. \quad \dots (6)$$

The above is a Quadrature formula involving $2n+2^n$ functional values. Putting $n=2$, we get Burnside's result.

2. If the degree of $f \geq 7$, we have to assume along with $g(a)$ and $h(b)$ as before, if $n \geq 2$,

$$k(c) = f(c, c, 0, 0, 0, \dots) + f(+c, -c, 0, 0, 0, \dots) \\ + f(-c, +c, 0, 0, 0, \dots) + f(-c, -c, 0, 0, 0, \dots) + \dots$$

a summation of $4 \cdot {}^nC_2$ terms.

Now assume,

$$\theta = a_{6000} \dots + a_{0600} \dots + \dots; \quad \phi = a_{4200} \dots + a_{0420} \dots + \dots; \\ \psi = a_{2202} \dots + a_{2022} \dots + a_{0222} \dots + \dots$$

so that
$$I = 2^n \left(\frac{\theta}{7} + \frac{\phi}{15} + \frac{\psi}{27} + \frac{\alpha}{5} + \frac{\beta}{9} + \frac{\gamma}{3} + \delta \right)$$

and,

$$\left. \begin{aligned} g(a) &= 2^n \{ a^6 (\theta + \phi + \psi) + a^4 (\alpha + \beta) + a^2 \gamma + \delta \} \\ h(b) &= 2 \{ b^6 \theta + b^4 \alpha + b^2 \gamma + n \delta \} \\ k(c) &= 4c^6 \{ (n-1)\theta + \phi \} + 4c^4 \{ (n-1)\alpha + \beta \} \\ &\quad + 4c^2 (n-1)\gamma + 2n(n-1)\delta. \end{aligned} \right\} \dots (7)$$

So, putting $\lambda g(a) + \mu h(b) + \nu k(c) + \rho \delta = I$, we get 7 equations to solve for $\lambda, \mu, \nu, \rho, a, b, c$ from which we get,

$$\left. \begin{aligned} \lambda &= \frac{1}{27a^6}; \quad \mu = \frac{2^n(64-14n)}{7 \times 135b^6}; \quad \nu = \frac{2^n}{135c^6}; \\ 2^n \lambda + 2^n \mu + 2n(n-1)\nu + \rho &= 2^n; \\ \frac{1}{27a^2} + \frac{2(64-14n)}{b^2 \times 7 \times 135} + \frac{4(n-1)}{135c^2} &= \frac{1}{5}; \\ \frac{1}{27a^4} + \frac{2(64-14n)}{b^4 \times 7 \times 135} + \frac{4(n-1)}{135c^4} &= \frac{1}{3}; \\ \frac{1}{27a^2} + \frac{4}{135c^2} &= \frac{1}{9}. \end{aligned} \right\} \dots (8)$$

From (8), it could be seen that real values of a, b, c are available for all integral values of $n \geq 0$.

So, we get the following result.

If f is an integral function of n variables, of degree not exceeding 7, for $n > 2$,

$$\begin{aligned} \int_{-1}^1 \int_{-1}^1 \dots \int_{-1}^1 f(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n \\ = \lambda \Sigma f(\pm a, \pm a, \pm a, \dots, \pm a) + \mu \Sigma f(\pm b, 0, 0, \dots, 0) \\ + \nu \Sigma f(\pm c, \pm c, 0, \dots, 0) + \rho f(0, 0, 0, \dots, 0). \end{aligned}$$

where $a, b, c, \lambda, \mu, \nu, \rho$ have values given by equations (8). ... (9)

Similarly we can prove the following general result.

Theorem: If the degree of f $\nless m$ (an odd integer) and $n > \alpha - 1$, then if $\frac{m^2 + 7}{8}$ is even and $= 2\sigma$,

$\int_{-1}^1 \int_{-1}^1 \dots \int_{-1}^1 f(x_1, x_2, \dots, x_n) dx_1 dx_2 \dots dx_n$, where f is an integral function of n variables, can be expressed as a linear function of

$\Sigma f(\pm a, \pm a, \dots, \pm a, \pm a), \Sigma f(\pm b, 0, 0, \dots, 0), \Sigma f(\pm c, \pm c, 0, 0, \dots, 0),$

$\Sigma f(\pm d, \pm d, \pm d, 0, 0, \dots)$, the number of such summations being σ ;

while if $n \nless \rho - 1$ and $\frac{m^2 + 7}{8}$ is odd and $= 2\rho + 1$ (ρ being integer), the

integral could be expressed as a linear function of the set of ρ functions of the type indicated for even case, taken along with $f(0, 0, 0, \dots, 0)$

...(10).

ON APOLAR QUADRICS

By

JAGDISH GOPAL ANAND

(Communicated by Prof. C. V. H. Rao)

I. Miss Telling has, in an article published in the "Proceedings of the Camb. Phil. Society, Vol. XXI, Part III," considered the construction of self-conjugate pentads and hexads for a non-degenerate quadric Σ . She has also proved that if the vertices of a figure of one of the three types — a self-polar tetrahedron, a self-conjugate pentad and a self-conjugate hexad w.r.t. Σ — can be found on a cubic σ , an infinite number of figures of each of the other two types can be found also having their vertices on σ . From this she has deduced the corresponding theorems for any quadric S . Those proofs mostly depend upon the methods of construction of these figures.

In this article only the properties of these figures are used and it is proved that if the vertices of a figure of one of these three types lie on a quadric S , S contains the vertices of an infinite number of each of the other two types of figures. In the end an interesting property of a cubic outpolar to Σ is added.

II. The following two well-known results are assumed:—

1. If S contains the vertices of one tetrahedron self-polar w.r.t. Σ , it contains the vertices of an infinite number of such tetrahedrons.

2. Any quadric S passing through seven of the eight vertices of two tetrahedrons self-polar w.r.t. any, the same quadric Σ passes through their eighth vertex also.

Let us now prove:—

3. If a quadric S circumscribes a pentad self-conjugate w.r.t. another quadric Σ , then an infinite number of tetrahedrons self-polar w.r.t. Σ can be inscribed in S .

Let $ABCDE$ be a pentad satisfying the given conditions. We can construct a tetrahedron $AB_1C_1D_1$ which is self-polar w.r.t. Σ such that its vertices B_1 and C_1 besides the vertex A lie on S .

There is a family ϕ of quadrics having ring contact with Σ along the plane $B_1C_1D_1$. This family has $AB_1C_1D_1$ as a common self-

polar tetrahedron. Let Σ' be that member of this family which has the points D, E as a pair of conjugate points.

The poles of any plane w.r.t. the family ϕ lie on a straight line. Consider the plane BCD. Its pole w.r.t. Σ lies on AE. Also the pole of BCD w.r.t. the point-square A regarded as a degenerate member of the family ϕ is A itself. Therefore AE is the line of poles of BCD w.r.t. the family ϕ . As the polar plane of D w.r.t. Σ' passes through E, it is clear that E is the pole of BCD w.r.t. Σ' . Similarly, it can be proved that D is the pole of BCE w.r.t. Σ' . Now from the fact that B, D and C, D have been proved to be pairs of conjugate points for Σ' , it follows similarly, that B and C are the poles w.r.t. Σ' of the planes CDE and BDE respectively, i.e., BCDE is a self-polar tetrahedron w.r.t. a quadric Σ' for which $AB_1C_1D_1$ is also a self-polar tetrahedron. As S passes through the points A, B_1 , C_1 , B, C, D and E, it must pass through the point D_1 also. This means that S circumscribes one, and therefore an infinite number of tetrahedrons, self-polar w.r.t. Σ' .

In a similar manner we can prove that if one and therefore an infinite number of tetrahedrons self-polar w.r.t. Σ are inscribed in S, an infinite number of pentads self-conjugate w.r.t. Σ can be inscribed in S.

4. If a quadric S circumscribes a self-conjugate hexad w.r.t. a quadric Σ , then an infinite number of tetrahedrons self-polar w.r.t. Σ can be inscribed in S.

Let ABCDEF be a hexad satisfying the given conditions. Again construct a tetrahedron $AB_1C_1D_1$ self-polar w.r.t. Σ having the vertices B_1 , C_1 , besides the vertex A on S. Consider the family ϕ of quadrics defined in §3. The line of poles of the plane DEF w.r.t. this family will pass through A and will lie in the plane ABC for the pole of DEF w.r.t. Σ lies in the plane ABC. Let Σ'' be defined as that quadric of the family ϕ for which the pole of DEF lies on BC.

The planes BCF and DEF are then conjugate for the quadric Σ'' . Also the line of poles of the plane BCF w.r.t. the family ϕ lies in the plane ADE. Therefore the pole of BCF w.r.t. Σ'' must lie on DE — the line common to the planes DEF and ADE.

Similarly, it can be shown that the poles of BCE and BCD w.r.t. Σ'' lie on DF and EF respectively.

From the fact that the pole of BCE w.r.t. Σ'' lies on DF, it would similarly follow that the poles of the planes DFB, DFC lie on CE and BE respectively.

It is clear now that any plane of the pentad BCDEF can be shown to have its pole on the opposite line. This means that the pentad

BCDEF is self-conjugate for Σ'' . From §3 therefore it follows that D_1 lies on S. Hence one, and therefore an infinite number of tetrahedrons self-polar w.r.t. Σ , can be inscribed in S.

Similarly we can prove the converse of this theorem.

5. Combining §§ 1, 2, 3, 4 we see that if the vertices of one figure of any of the three types—a self-polar tetrahedron, a self-conjugate pentad and a self-conjugate hexad w.r.t. Σ —can be found on S, the vertices of an infinite number of figures of the other two types can also be found on S.

III. A property of outpolar cubics.

Let us suppose that a cubic σ contains the vertices of a self-conjugate pentad or hexad w.r.t. a quadric Σ . From any point P outside σ draw the chord l to σ . Two independent quadrics S and S' can be made to pass through σ and l . The polar plane w.r.t. Σ of any point common to S and S' would meet S and S' in conics which are both outpolar to the conic in which that plane meets Σ . In particular this result is true about the point P. If the polar plane π of P w.r.t. Σ meets S and S' in the conics c and c' , the four common points of c , c' form a self-conjugate 4-set for the section of Σ by π . These are the four points in which π is met by σ and l .

Therefore the chord through any point P to σ meets the polar plane π of P w.r.t. Σ in a point which with the three points in which π meets σ forms a self-conjugate 4-set of points for the section of Σ by π .

Let the point P now take up a position on the cubic σ . Then through P can be drawn an infinite number of chords of the cubic σ . Therefore there are an infinite number of points in the polar plane π of P w.r.t. Σ which with the points of intersection of σ with π form a self-conjugate 4-set for the section of Σ by π . This means that the polar plane of P meets σ in a self-polar triangle for Σ or that an infinite number of tetrahedrons self-polar w.r.t. Σ can be found having their vertices on σ .

Hence if on a cubic σ can be found the vertices of a self-conjugate pentad or hexad w.r.t. Σ , σ circumscribes an infinite number of tetrahedrons self-polar w.r.t. Σ .

ON AFFINE REGULARITY OF POLYGONS

By

MISS ABHA MITRA

(Communicated by Dr. F. W. Levi)

J. Douglas has given an interesting theory of the metric properties of polygons in his "Geometry of Polygons in the Complex plane."* By the help of this theory, we can establish some other geometrical properties of affine regular polygons in an algebraic manner. It can be shown in a purely geometrical manner, that a pentagon is affine regular, if and only if, every diagonal is parallel to a side. In the following lines it will be proved that this theorem can be extended to a theorem, applicable to every polygon.

§ 1. In this section some definitions and results of Douglas will be given. The geometrical entities are expressed by numbers of the complex $z = x + yi$ -plane throughout this paper.

Definition 1. An n -gon $(z_1, z_2, \dots, z_n)$ is said to be affine regular of ω^p -type, if it is similar to the n -gon $(1, \omega^p, \omega^{2p}, \dots, \omega^{(n-1)p})$,

where
$$\omega = e^{\frac{2\pi i}{n}}; p = 1, 2, \dots, n-1.$$

For $p=1$ or $n-1$, the regular polygons are convex and the same but for orientation; for other values of p which are prime to n , the polygons are star polygons. If, however, $n=rs$, we have a regular r -gon, described s times in succession, and the r -gon may be convex or star-shaped.

Definition 2. An n -gon $(z_1, z_2, \dots, z_n)$ is said to be affine regular of ω^p -type, if it is obtainable from the n -gon $(1, \omega^p, \omega^{2p}, \dots, \omega^{(n-1)p})$ by some affinity transformation.

Douglas has stated the following $n-1$ relations, called R' -relations, with regard to an n -gon $(z_1, z_2, \dots, z_n)$,

$$R'_q : z_1 + \omega^q z_2 + \omega^{2q} z_3 + \dots + \omega^{(n-1)q} z_n = 0$$

$q=1, 2, \dots, n-1$ where $\omega = e^{\frac{2\pi i}{n}}$, $s_k = z_{k+1} - z_k$; and has shown that

* Journal of Mathematics and Physics, vol. XIX, p. 98.

(a) An n -gon reduces to a point, if and only if, the sides $s_1, s_2, \dots, s_n$ satisfy all the R' -relations ;

(b) An n -gon is regular of ω^p -type, if and only if, the sides satisfy all the R' -relations except $R'_p=0$;

(c) An n -gon is affine regular of ω^p -type, if and only if, the sides satisfy the R' -relations, except possibly $R'_p=0$ and $R'_{n-p}=0$.

§ 2. Let 1, 2, ..., n be the vertices of an n -gon, $z_1, z_2, \dots, z_n$ being the corresponding complex numbers and let the sides $z_{k+1}-z_k$ be denoted by s_k . Let the vertices and the sides be determined mod. n . We shall establish the following theorem :

Theorem. *The n -gon is affine regular, if and only if,*

- (i) *the $n > 2$ vertices are non-collinear,*
- (ii) *every side is parallel to a 3-diagonal **
- and (iii) *every 2-diagonal is parallel to a 4-diagonal.*

The conditions are obviously necessary, as they hold in any regular polygon and they are not altered by affinity. We prove the converse proposition. Suppose the n -gon to satisfy the "conditions of parallelism", namely the following equations :

$$s_k + s_{k+2} = (\lambda_{k+1} - 1) s_{k+1}, \quad \dots \quad (1)$$

$$s_k + s_{k+3} = (\mu_{k+1} - 1)(s_{k+1} + s_{k+2}), \quad \dots \quad (2)$$

$k=1, 2, \dots, n$ and λ_{k+1} and μ_{k+1} are real numbers.

Consider the following three equations of (1) and (2):

$$\left. \begin{aligned} s_1 + (1 - \lambda_2) s_2 + s_3 &= 0 \\ s_2 + (1 - \lambda_3) s_3 + s_4 &= 0 \\ s_1 + (1 - \mu_2) s_2 + (1 - \mu_2) s_3 + s_4 &= 0. \end{aligned} \right\} \quad \dots \quad (3)$$

Now a system of equations in complex variables with real coefficients is equivalent to a pair of systems of equations in real variables, the coefficients being exactly the same in the two systems. We write $s_j = a_j + b_j i$ and the numbers a_j and b_j are the solutions of the same system of linear homogeneous equations with the real matrix

$$\left\| \begin{array}{cccc} 1 & 1 - \lambda_2 & 1 & 0 \\ 0 & 1 & 1 - \lambda_3 & 1 \\ 1 & 1 - \mu_2 & 1 - \mu_2 & 1 \end{array} \right\| \quad \dots \quad (4)$$

* A 2-diagonal is that one which joins k to $k+2$ and a 3-diagonal is that which joins k to $k+3$ and so on. Hence in a quadrangle every side is simultaneously a 3-diagonal.

If (4) has only one solution, then a_j is the same multiple of b_j for every j , and the vertices of the n -gon are then collinear. Hence, there must be at least two independent solutions, the rank of the matrix (4) is, therefore, at most two. Hence

$$\begin{vmatrix} 1 & 1-\lambda_2 & 0 \\ 0 & 1 & 1 \\ 1 & 1-\mu_2 & 1 \end{vmatrix} = 0$$

whence $\mu_2 + 1 - \lambda_2 = 0$ or $\mu_2 = \lambda_2 - 1$.

By equating to zero another suitable determinant we get

$$\mu_2 = \lambda_3 - 1.$$

Therefore $\lambda_2 = \lambda_3$.

By cyclic permutations one gets

$$\lambda_1 = \lambda_2 = \lambda_3 = \dots = \lambda_n = \lambda, \text{ say.}$$

Hence also $\mu_1 = \mu_2 = \mu_3 = \dots = \mu_n = \mu, \text{ say.}$

We may, therefore, replace the conditions (1) and (2) by

$$\rho s_k + s_{k+1} + s_{k-1} = 0, \quad k=1, 2, \dots, n, \quad \dots (5)$$

where $\rho = 1 - \lambda$.

Hence, we need to see the nature of the n -gon whose sides satisfy (5) together with the additional relation $\sum_{j=1}^n s_j = 0$. Excluding the value -2 of ρ , because only in that case $\sum_{j=1}^n s_j = 0$ does not follow from (5), we restrict our consideration to the equations (5) only. Indeed, when $\rho = -2$, λ becomes equal to 3 $\lambda = 3$, and the points are collinear.

If now there exists a system s_j , satisfying (5), then ρ must be a root of

$$\begin{vmatrix} \rho & 1 & 0 & 0 & \dots & \dots & \dots & \dots & 0 & 1 \\ 1 & \rho & 1 & 0 & \dots & \dots & \dots & \dots & 0 & 0 \\ 0 & 1 & \rho & 1 & 0 & \dots & \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & \dots & \dots & \dots & \dots & 0 & 1 & \rho \end{vmatrix} = 0; \quad \dots (6)$$

and, conversely, for every value of ρ satisfying (6), there are solutions of (5). This determinant again can be put into the factors*

$$(\rho+2)(\rho+\omega+\omega^{n-1})(\rho+\omega^2+\omega^{n-2}) \dots (\rho+\omega^{n-1}+\omega)$$

which can be written as

$$(\rho+2)(\rho+\omega+\omega^{-1})^2(\rho+\omega^2+\omega^{-2})^2 \dots (\rho+\omega^{\frac{n-1}{2}}+\omega^{\frac{n+1}{2}})^2$$

when n is odd, and as

$$(\rho+2)(\rho+\omega+\omega^{-1})^2(\rho+\omega^2+\omega^{-2})^2 \dots (\rho+\omega^{\frac{n}{2}-1}+\omega^{\frac{n}{2}+1})^2 (\rho+\omega^{\frac{n}{2}}+\omega^{-\frac{n}{2}})$$

when n is even. Hence ρ , under consideration, is given by

$$\rho = -(\omega^p + \omega^{-p}), \quad p = 1, 2, \dots, n-1,$$

of which only $(n-1)/2$ values are distinct when n is odd, and $n/2$ values, when n is even.

Take any particular value of ρ , say $-(\omega^p + \omega^{-p})$, then we shall see that the $n-3$ relations

$$R_q: s_1 + \omega^q s_2 + \dots + \omega^{(n-1)q} s_n = 0$$

$q=1, 2, \dots, n-1$ but $\neq p$ and $n-p$ are linearly dependent on (5). The question, therefore, rests upon whether it is possible to solve the following equations for $m_j=1, 2, \dots, n$:

$$\rho m_1 + m_2 + m_n = 1$$

$$m_1 + \rho m_2 + m_3 = \omega^q$$

$$m_2 + \rho m_3 + m_4 = \omega^{2q}$$

$$\dots \dots \dots$$

$$m_1 + m_{n-1} + \rho m_n = \omega^{(n-1)q}.$$

This will be possible if the matrix

$$\left\| \begin{array}{cccccccc} \rho & 1 & 0 & \dots & \dots & \dots & 1 & 1 \\ 1 & \rho & 1 & 0 & \dots & \dots & 0 & \omega^q \\ 0 & 1 & \rho & 1 & 0 & \dots & 0 & \omega^{2q} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & \dots & 0 & 1 & \rho & \omega^{(n-1)q} \end{array} \right\| \dots \quad (7)$$

* The Theory of Equations by Burnside and Panton, Vol. II, p. 62.

has the same rank as the matrix, obtained by dropping the last column. Now from the t^{th} column, $t=1, 2, \dots, n$

$$\omega^{(t-2)p} + \rho \omega^{(t-1)p} + \omega^{tp} = \omega^{(t-1)p} (\omega^{-p} + \rho + \omega^p) = 0.$$

So denoting the row-vectors of the matrix of the first n columns of (7) by $a^1, a^2, \dots, a^n$, we get

$$\left. \begin{aligned} \alpha^1 + \omega^p \alpha^2 + \omega^{2p} \alpha^3 + \dots + \omega^{(n-1)p} \alpha^n &= 0 \\ \alpha^1 + \omega^{-p} \alpha^2 + \omega^{-2p} \alpha^3 + \dots + \omega^{-(n-1)p} \alpha^n &= 0. \end{aligned} \right\} \dots \quad (8)$$

Consequently, two of the α 's are linearly dependent on the other $(n-2)$ α 's. Hence the rank of the second matrix cannot be greater than $n-2$. That this rank is exactly $n-2$ can be seen from the fact that, the value of the determinant, formed by the first $n-2$ rows and the second $n-2$ columns, is equal to one. Again since both of

$$1 + \omega^{q+p} + \omega^{2(q+p)} + \dots + \omega^{(n-1)(q+p)} = 0$$

$$1 + \omega^{q-p} + \omega^{2(q-p)} + \dots + \omega^{(n-1)(q-p)} = 0$$

hold for all values of q but not for $q=p$ or $-p$, the last column of (7) is linearly related in the same ways as the other columns. Hence the $(n-5)$ R'_2 -relations for $q=1, 2, \dots, n-1$ but $\neq p$ and $n-p$, follow from (5). Thus the theorem is proved.

We next investigate how many and which ones of (1) and (2) are independent. We can safely omit one of the conditions of (1) and two of the conditions of (2). For instance, suppose the first $n-1$ conditions of (1) and the first $n-2$ conditions of (2) hold. Then we can prove, as before, that

$$\lambda_2 = \lambda_3 = \dots = \lambda_n = \lambda \text{ (say) and } \mu_2 = \mu_3 = \dots = \mu_{n-1} = \mu \text{ (say),}$$

so that the above $2n-3$ conditions can be replaced by

$$\left. \begin{aligned} s_1 + \rho s_2 + s_3 &= \dot{0} \\ s_2 + \rho s_3 + s_4 &= \dot{0} \\ \dots &\dots \\ s_1 + \dots + s_{n-1} + \rho s_n &= 0 \end{aligned} \right\} \dots (9)$$

where $\rho = 1 - \lambda$.

Adding these and remembering that $\sum_{j=1}^n s_j = 0$, we get $\rho s_1 + s_2 + s_n = 0$ which is the last equation of (1); adding the first $n-2$ relations of (9) we get the $(n-1)^{\text{th}}$ condition of (2). On the other hand, there exist

pentagons in which exactly three conditions are satisfied ; we cannot, therefore, omit any more conditions of (1).

We note that in any particular case the number of independent conditions may reduce to a number less than $2n-3$. For instance, in the case of a pentagon conditions (2) become identical with conditions (1), hence a pentagon is affine regular, if and only if, four of the sides are parallel to four diagonals. In the case of a hexagon the six conditions (2) reduce to three conditions, so any five conditions from (1) and any two from (2) will suffice for the hexagon.

It may be interesting to see the nature of the hexagon which satisfies (1) only, namely

$$s_2 + \rho_1 s_1 + s_6 = 0 \quad (i) \qquad s_3 + \rho_4 s_4 + s_5 = 0 \quad (iv)$$

$$s_1 + \rho_2 s_2 + s_3 = 0 \quad (ii) \qquad s_4 + \rho_5 s_5 + s_6 = 0 \quad (v)$$

$$s_2 + \rho_3 s_3 + s_4 = 0 \quad (iii) \qquad s_1 + \rho_6 s_6 + s_5 = 0 \quad (vi)$$

Adding (i) and (iv) one gets $s_1 + \sigma_4 s_4 = 0$ (vii), where $\sigma_4 = \frac{\rho_4 - 1}{\rho_1 - 1}$

while (ii) and (v) give $\sigma_2 s_2 + s_5 = 0$ (viii), where $\sigma_2 = \frac{\rho_2 - 1}{\rho_5 - 1}$

and (iii) and (vi) give $s_3 + \sigma_6 s_6 = 0$ (ix), where $\sigma_6 = \frac{\rho_6 - 1}{\rho_3 - 1}$.

$$\text{Put} \quad -\rho_5 \sigma_2 = k_2 \quad -\rho_1 \sigma_4 = k_4 \quad -\rho_3 \sigma_6 = k_6.$$

From (i), (iii), (v), (vii), (viii) and (ix)

$$k_2 s_2 + s_4 + s_6 = 0$$

$$s_2 + k_4 s_4 + s_6 = 0$$

$$s_2 + s_4 + k_6 s_6 = 0.$$

For reasons, as explained before, the matrix

$$\begin{vmatrix} k_2 & 1 & 1 \\ 1 & k_4 & 1 \\ 1 & 1 & k_6 \end{vmatrix}$$

must be of rank one, whence one gets $k_2 = k_4 = k_6 = 1$.

$$\text{So} \quad s_2 + s_4 + s_6 = 0.$$

Add (i), (iii), (v), and from the sum subtract $2(s_2 + s_4 + s_6) = 0$, then

$$\rho_1 s_1 + \rho_3 s_3 + \rho_5 s_5 = 0,$$

$$s_1 + s_3 + s_5 = 0.$$

The rank of the matrix of coefficients will be equal to one, so one gets

$$\rho_1 = \rho_3 = \rho_5.$$

Hence

$$\rho_1 = \rho_3 = \rho_5 = -\frac{\rho_1 - 1}{\rho_4 - 1} = -\frac{\rho_1 - 1}{\rho_2 - 1} = -\frac{\rho_1 - 1}{\rho_6 - 1}$$

hold in the hexagon, under consideration.

Here I take the opportunity of thanking Dr. F. W. Levi, Hardinge Professor under whose guidance this work has been done and Mr. R. C. Bose, Lecturer of our department, for his kind help.

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DETERMINATION OF THE GROUPS OF ORDER 180

By

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(Communicated by Prof. F. W. Levi)

Lately¹ Senior and Lunn have found out a method of determining all the soluble groups of a particular order (finite). It is based on the formation of intransitive groups by combination of permutation groups according to different isomorphisms. The other method depends on Schreier's extension theorem and on the formation of transitive permutation groups. Senior and Lunn have given symbolic notation for all the soluble groups. In the following pages it has been tried to find out an interrelation between the two methods.

That the solubility of the group is essential in the method of Senior and Lunn follows from the following theorem due to Hall²:

If G is any soluble group of order yz , where y and z are relatively prime, i.e., $(y, z)=1$, then

- (a) G contains at least one subgroup of order y ,
- (b) all subgroups of order y are conjugate,
- (c) every subgroup whose order divides y is contained in a subgroup of order y .

So if a group G is of order $g=\pi x_i$, where the x_i 's are powers of different primes, then for each i , a complete set of conjugate subgroups of index x_i exists in G and so a transitive permutation group T_i of order kx_i can be formed such that $T_i \cong G/C_i$, where C_i is the meet of all such T_i 's. If R_i is an abstract group isomorphic to T_i , then a complete array of R 's can be found and combined by isomorphisms so as to give a group of order g . For a given G there exists one and only one such isomorphism, and only one complete array of R 's to be combined. Thus picking up different array of R 's and different isomorphisms one gets all possible groups of a given order.

In Schreier's Theorem method one has to consider the number of Sylowgroups corresponding to a particular prime number. If the number be only one, one applies Schreier's extension

¹ American Journal of Mathematics, Vol. 56.

² Journal of the London Mathematical Society, Vol. 3.

theorem.¹ If the number be $r > 1$ then $G/H \cong$ a transitive permutation group of index r , where H is the meet of the normalisers of all the r Sylowgroups and one can get all the groups of this type by Schreier's method.

In this paper an enumeration of the groups of order 180 has been made (§ 1) by Schreier's extension theorem; the groups are represented by generators and relations. Furthermore the interconnection between the two methods has been considered (§ 2).

I here take the opportunity of expressing my thanks to Mr. P. Hall, of King's College, Cambridge, who asked me to find out all the groups of this particular order by Schreier's method and to my esteemed professor Dr. F. W. Levi of Calcutta University, for his guidance and assistance in completing the paper in its present form.

§ 1. ENUMERATION BY SCHREIER'S METHOD

A group G of order 180 may possibly have 36, 6 or 1 Sylowgroups of order 5. Each case will be considered separately.

1. There is no group of order 180 with 36 Sylowgroups of order 5.

Suppose G has 36 Sylowgroups of order 5. Then G is homomorphic to a transitive permutation group on 36 symbols. This permutation group cannot be more than singly transitive, *i.e.*, there is no permutation leaving two symbols a_1, a_2 unchanged. Because, if there be such a permutation α and if H_1 and H_2 be the subgroups leaving a_1 and a_2 respectively fixed, then α would belong to $H_1 \wedge H_2$. But H_1 and H_2 are each of order 5, hence the meet is the identical permutation. So α cannot be different from 1. Such a permutation group of degree n has² exactly $n-1$ permutations which displace all the n symbols and these $n-1$ permutations together with the identity always form a normal subgroup.² Since no permutation leaves two symbols unchanged, the subgroup H that leaves one symbol invariant must permute the remaining $n-1$ symbols and is intransitive. Hence if nm be the order of the group G , then H permutes the remaining $n-1$ symbols in set of m , so that m is a factor of $n-1$ and is prime to n . Because, if H_a be a transitive consti-

¹ H. Zassenhaus: Lehrbuch der Gruppentheorie. Vol. 1 Kap. III. §6. See Schreier's papers in Monatshefte für Math. u. Phys. Bd. 24 and Hamb. Abh. 4, p. 321.

² Burnside: Theory of Groups. 2nd Edition. §§193-34. Frobenius: Über auflösbare Gruppen IV.

tuent on α symbols, h_a be the normal subgroup that changes these α symbols only, γ_a be the normal subgroup which keeps all the α symbols unchanged, then the order¹ n_H of $H = n_{H_a} n_{\gamma_a}$. In this case $n_{\gamma_a} = 1$. Hence the order of each transitive constituent of H is the same as that of H . So the symbols of H are divided into sets of $n_H = m$.

The orders of the permutations which leave one symbol unchanged are factors of m . Hence the $n-1$ permutations which displace all the symbols and the identical permutation are all those satisfying $S^n = 1$. No permutation which leaves one symbol unchanged can be permutable with any permutation which displaces all the symbols, hence the number of those which belong to any conjugate set² is a multiple of m .

Let p^a be the highest power of a prime p which divides n , so that $(m, p) = 1$. The number of permutations whose orders divide n/p^a is a multiple of this number,³ say $k_p n/p^a$; ($k_p < p^a$). k_p cannot be equal to p^a , because in that case all the n permutations would be of order prime to p , which is contrary to Sylow's theorem. The number of permutations whose orders are multiples of p is therefore $n - k_p n/p^a$. This number must be a multiple of m . Hence, since $(n, m) = 1$, m is a factor of $p^a - k_p$, where $k_p < p^a$.

Applying the above in case of the group of order 180, we see, by taking $p=2$, $m=5$, that 5 divides $2^2 - k_2$ for some value of $k_2 < 4$. This is impossible. Hence there cannot⁴ be such a group of order 180 with 36 Sylowgroups of order 5.

2. There is exactly one group of order 180 with 6 Sylowgroups of order 5.

Let G have 6 Sylowgroups of order 5. Then, if H is the meet of the normalisers of these 6 Sylowgroups, then $G/H \cong$ a transitive permutation group of degree 6. Such a permutation group G' should either be primitive or imprimitive. But there are⁵ only two types of primitive groups of degree 6, and these are of order 120 and 60. If G' be imprimitive there would be either 2 or 3 systems of imprimitivity and G' would be homomorphic to a group of degree 2 or 3,

¹ Burnside : *loc. cit.* §144. Also §2 of this paper.

² Burnside : *loc. cit.* § 184.

³ Carmichael : Groups of Finite order. §22. Also Burnside, *loc. cit.* §48.

⁴ Miller, Blichfeldt and Dickson : Theory and Applications of Finite Groups. Chap. VIII. §74.

⁵ Burnside : *loc. cit.* §166.

so it would contain a normal subgroup of order 90 or 60 (or a multiple of 60). The permutations of this normal subgroup would permute the symbols of each of the set of imprimitivity among themselves¹; hence the order of the normal subgroup cannot exceed 31. So there cannot be any imprimitive group of degree 6 and order 180. Hence the order of H must be >1 , and it must be 2, 3 or 6. (5, 10 and 15 are not tenable since in these cases H would contain a Sylowgroup of G which is impossible.)

Before considering the exact order of H , the solubility of G will be considered. In a soluble group the number of Sylowgroups of order p^n is always a product of prime powers, each of which $\equiv 1 \pmod{p}$. In this case G has only 6 Sylowgroups of order 5, which does not satisfy the above condition. Hence all such groups are insoluble. Since G is insoluble, H being normal, G/H is also insoluble. But there is no insoluble group of order 90 or 30, the only insoluble group of order less than 100 is² of order 60. Hence H is of order 3 and $G/H \cong$ insoluble group of order 60, *i.e.*, the icosohedral group. Since G contains 6 Sylowgroups of order 5, which generate $\bar{G}$, where $\bar{G}H = G$, so H centralises $\bar{G}$ as well as itself; hence $H \subset Z(G)$, the central of G ; again since G/H is simple $H \supset Z(G)$. Therefore $H = Z(G)$.

From Schreier's theorem³ it follows that there exists a sub-group of G , $\cong G/H$, *i.e.*, an icosohedral group. H then centralises such a sub-group and G is the direct product of H and an icosohedral group. Thus there is only one group of order 180 with 6 Sylowgroups of order 5.

3. There are 36 non-isomorphic groups of order 180 with one sub-group of order 5. Before considering this case a brief demonstration of Schreier's extension theorem will be given.

SCHREIER'S EXTENSION THEOREM⁴

Let H and K be subgroups of G , such that $H \wedge K = 1$, and $HK = G$. Then H and K are said to be complementary in G and each of these subgroups is called the complement of the other. If H is normal in

¹ Burnside : *loc. cit.* §143.

² Hölder : *Mathematische Annalen*, Vol. 46.

³ Schreier : *Monatshefte für Math. u. Phys.*, Bd. 34.

⁴ Zassenhaus : *loc. cit.*

G , then $G/H \cong K$. Every element in G is uniquely expressible in the form $h_1 k_1$, where h_1 is an element of H , and k_1 is an element of K . From $K \cong G/H$ the product of any two such elements $(h_1 k_1)(h_2 k_2) = h_1 (k_1 h_2 k_1^{-1}) k_1 k_2$ can be obtained; $k_1 k_2$ is known from the structure of $K \cong G/H$ and $k_1 h_2 k_1^{-1}$ is an element of H and is known if the automorphisms of H which are associated with the elements of K are known. Hence given a normal subgroup H and its quotient-group $K \cong G/H$, the structure of G is uniquely determined when the automorphisms of H associated with the elements of K are known. If θ_k be an automorphism of H corresponding to an element k of K , i.e., to $k^{-1} h k$, then the correspondence $k \rightarrow \theta_k$ gives a homomorphism of K onto a subgroup of the group of automorphisms of H . So in order to find out the structure of G the above homomorphism will have to be considered.

Conversely, given H , G/H and any homomorphism of G/H onto a subgroup of the group of automorphisms of H , there exists a corresponding group G which contains H as a normal subgroup and in which H is complemented.

In the present case H is of order 5, G/H of order 36. The group of automorphisms of H is of order 4 and is cyclic. Different structures of G can be determined by considering different groups of order 36 and also different homomorphisms of these groups of order 36 onto a subgroup of the cyclic group of order 4. So before determining all the groups of order 180 the groups of order 36 will be obtained and represented by generators and relations. That there are 14 essentially different groups of order 36 has already been stated by G. A. Miller.²

Groups of order 36

These groups will be classified according to the number of Sylow-groups of order 9. There may be 4 or 1 such Sylowgroups.

(a) If the group G_1 of order 36 has 4 Sylowgroups of order 9, then there exists a transitive permutation group $\cong G_1/H$ on 4 symbols, where H is the meet of the normalisers of the 4 Sylowgroups. H cannot be of order 1, because GS_4 (the symmetric group on 4 symbols) has no subgroup of order 36. The order of the normalisers of the Sylowgroups are also 9, i.e., the Sylowgroups are their own

¹ Andreas Speiser : Die Theorie der Gruppen von endlicher Ordnung 2nd Edition. Theorem 24 and 25.

² American Journal of Mathematics, Vol. 52.



normalisers, so H belongs to all the Sylowgroups of order 9 and is of order 3. So $G_1/H \cong$ a transitive permutation group of order 12, i.e., the alternating group of degree 4. Since H is of order 3, let $1, p, q=p^2$ be the elements of H . Then p, q are either conjugate in the group G_1 or they are each self-conjugate (H being normal). The Sylowgroups being their own normalisers and Abelian, p and q belong to the central of each Sylowgroup and they are conjugate in G_1 if and only if¹ they are conjugate in the Sylowgroups. Hence they cannot be conjugate in G_1 . So they are self-conjugate in G_1 or $H \subset Z$, the central of G_1 . The tetrahedral group has the following generating relations:—

$$s^2=t^2=u^3=1, \\ st=ts, \quad u^{-1}su=t, \quad u^{-1}tu=st,$$

and it has no proper central. Hence G_1/H has the central 1, i.e., $Z \subset H$. $\therefore H=Z$. Hence, in this case, each element of the tetrahedral group G_1/H correspond to the identical automorphism of H . So G_1 is the direct product of the tetrahedral group and a group of order 3. Since the tetrahedral group has a subgroup of order 3, there are 2 cases—when H and the subgroup of order 3 in the tetrahedral group generate a cyclic group of order 9 and when they generate an elementary Abelian group of order 9. Thus the two groups in this case are

$$A_1) \quad s^3=t^2=r^2=1, \\ rt=tr, \quad s^{-1}ts=r, \quad s^{-1}rs=tr.$$

$$A_2) \quad s_1^3=s_2^3=t^2=r^2=1, \\ s_1s_2=s_2s_1, \quad tr=rt, \quad s_2t=ts_2, \quad s_2r=rs_2, \\ s_1^{-1}ts_1=r, \quad s_1^{-1}rs_1=tr.$$

(b) Groups with one normal subgroup H of order 9.

(1) H : cyclic. The group of automorphisms of the cyclic group of order 9 is a cyclic group² of order $\phi(9)=6$. G_1/H is of order 4 and may be cyclic or Vierer. Since the group of automorphisms of H does not contain any element of order 4, the elements of order 4 (G_1/H is cyclic) or of order 2 (G_1/H is Vierer) will correspond to the automorphism of order 2 or 1 each. In the latter case, the group G_1 is the direct product of the two groups H and C_4 (cyclic group of order

¹ Speiser : *Loc. cit.* § 23.

² Speiser : *Loc. cit.* § 18. Die Galoisfelder und Reste nach Primzahlpotenz.

4) or V_4 (Vierer group) as the case may be. Thus the four groups in this case are

$$A_3) \quad C_9 \times C_4.$$

$$A_4) \quad C_9 \times V_4.$$

$$A_5) \quad s^9 = t^4 = 1, \quad t^{-1}st = s^{-1}.$$

$$A_6) \quad s^9 = t^2 = r^2 = 1, \quad tr = rt, \quad tst = s^{-1}, \quad rs = sr.$$

(2) H : elementary Abelian group of order 9. A homomorphism of the group of order 4 onto a subgroup of the group of automorphisms of H shall have to be found out. The group A of the automorphisms of H is¹ of order $(9-1)(9-3)=48$. The group A can be represented as a transitive permutationgroup on 8 symbols (the elements of order 3). The order of the subgroup that keeps one symbol fixed is 6 and it is the symmetric group of degree 3. H has got four subgroups of order 3, and by any automorphism these subgroups will be interchanged among themselves, i.e., by any automorphism such a subgroup will either be transformed into itself (so that the two elements will be interchanged) or will be transformed into some other (i.e., the two elements will be transformed into 2 other elements). Thus the 8 elements of order 3 fall into 4 sets of 2 each, such that by any automorphism, a system is either transformed into itself or into any other system. Hence the corresponding permutationgroup is imprimitive, having 4 systems of imprimitivity, each system containing an element and its inverse. The normal subgroup formed of those permutations that permute the symbols in each system among themselves is of order 2, and is the central Z of the group A . A/Z is of order 24 and is $\cong$ transitive permutation group of degree 4. Hence $A/Z \cong GS_4$, the symmetric group of degree 4.

GS_4 has² 3 octic Sylowgroups and 4 Sylowgroups of order 3. The three octic groups have a Vierergruppe in common and this Vierergruppe V contains elements of type (2^2) .

To every subgroup H' of GS_4 there corresponds a subgroup $\{H', Z\}$ of A of double the order. Thus there are 3 subgroups of order 16 with a common subgroup of order 8, generated by $\{V, Z\}$. Hence $\{V, Z\}$ contains 3 cyclic groups of order 4, so it is the quaternion group Q_8 .

¹ Speiser: *Loc. cit.* Kap. 9. § 48.

² Miller, Blichfeldt and Dickson: *Loc. cit.* § 2.

The elements of type (4) in GS_4 correspond to the elements of order 8 in A , and those of type (1²2) to (2). If α is an automorphism of order 8 then α^2 lies in $M=Q_8$ and M has 3 subgroups of order 4 containing $\{\beta\}=Z$, all of which are conjugate in A . Corresponding to the three subgroups of index 2 in the octic group there are 3 subgroups of index 2 in the Sylowgroups S of order 16 in A . These are M , the cyclic group $\{\alpha\}$ and the remaining one can be found from the following consideration. If L is that subgroup of order 8, it will be isomorphic to a permutationgroup on 4 symbols, s, s^2, t, t^2 if $H=\{s, t\}$. Hence $L \cong$ a subgroup of GS_4 , i.e., octic. Thus the 3 subgroups of index 2 are M (quaternion), $\{\alpha\}$ (cyclic) and L (octic). Since M contains α^2 , M and $\{\alpha\}$ has only one element of order 2; M has 3 cyclic subgroups of order 4, all conjugate in A and $\{\alpha\}$ has only one cyclic subgroup of order 4 which must lie in A . Thus A has a single class of cyclic groups of order 4. The homomorphism in this case can only be with a cyclic group of order 4, an element of order 4 either corresponds to an automorphism of order 4 or 1. The two groups are

$$\begin{aligned} & \cdot A_7) \\ & s^3 = t^3 = r^4 = 1, \\ & rs = sr, tr = rt, st = ts. \end{aligned}$$

$$\begin{aligned} & A_8) \\ & s^3 = t^3 = r^4 = 1, \\ & r^{-1}sr = t, r^{-1}tr = s^2. \end{aligned}$$

A has again a single class of V_4 , the subgroups in L . The groups in this case are

$$\begin{aligned} & A_9) \\ & s^3 = t^3 = r^2 = u^2 = 1, \\ & rsr = s^{-1}, rtr = t^{-1}, usu = s^{-1}, ut = tu. \end{aligned}$$

$$\begin{aligned} & A_{10}) \\ & E_9 \times V_4, \\ & E_9 = \text{elementary Abelian group of order 9.} \end{aligned}$$

L has 5 elements of order 2, one of which is $\{\beta\}$, four others are $\gamma_1, \gamma_1\beta, \gamma_2, \gamma_2\beta$ conjugate in pairs in L . γ_1, γ_2 are conjugate in A , because $\{\gamma_1, \beta\}, \{\gamma_2, \beta\}$, are so. Hence altogether there are 2 distinct classes of conjugate subgroups of order 2 in A , viz. $\{\beta\}$ and the class to which γ_1 belongs.

Thus 4 groups of order 36 are obtained by using automorphism of order 2 on an elementary Abelian group of order 9.

$A_{11})$

$$s^3 = t^3 = r^4 = 1, \\ r^{-1}sr = s^{-1}, r^{-1}tr = t^{-1}.$$

$A_{12})$

$$s^3 = t^3 = r^4 = 1, \\ r^{-1}sr = s^{-1}, rt = tr.$$

$A_{13})$

$$s^3 = t^3 = r^2 = u^2 = 1, \\ rsr = s^{-1}, rtr = t^{-1}, us = su, ut = tu, ru = ur.$$

$A_{14})$

$$s^3 = t^3 = r^2 = u^2 = 1, \\ rsr = s^{-1}, rt = tr, us = su, ut = tu, ru = ur.$$

So there are 14 groups of order 36. 6 of these $A_3, A_5, A_7, A_8, A_{11}, A_{12}$ with cyclic Sylowgroups of order 4 and the remaining 8 have Vierergruppen, as Sylowgroup.

Representation of the groups G_{180} of order 180 with one normal subgroup of order 5

The group of automorphisms of a group of order 5 is a cyclic group of order 4, and, to determine all the groups of order 180, one has to find out the different possible homomorphisms between the above 14 groups of order 36 and subgroups of the cyclic group of order 4. The homomorphism can only be established between the Sylowgroups of the groups of order 36 and the cyclic group of order 4. Those groups of order 36 which have cyclic Sylowgroups of order 4 give rise to the following homomorphisms:

- (1) The elements of order 4 may correspond to the identical automorphism
- (2) automorphism of order 2,
- (3) of order 4,

where the group of automorphisms is the cyclic group of order 4.

So there are 18 groups of order 180 which have cyclic Sylowgroups of order 4. The other 8 groups have Vierer Sylowgroups. The consideration of these 8 groups will be divided into 3 parts:

(a) If the two generating elements of the Vierergruppe have the same relation with the other elements of the group of order 36 (these are A_4 and A_{10})

then, either

- (1) both the elements correspond to the identical automorphism
or (2) one element corresponds to the identical automorphism
and the other to an automorphism of order 2.

So there are 4 groups of order 180 in this case.

(b) If the generating elements have different relations (as in A_6 , A_9 , A_{13} and A_{14}), one more homomorphism is obtained by interchanging the elements in case (a) (2).

So there will be 12 groups of order 180 in this case.

(c) The other 2 groups have each a Vierer normal subgroup and the case (2) is not applicable in this case, since these groups are

$$\begin{aligned} A_1) \quad & s^9 = t^2 = r^2 = 1, \\ & s^{-1}ts = r, \quad s^{-1}rs = tr. \end{aligned}$$

and A_2).

If now r corresponds to the automorphism of order 2, then

$$r^{-1}Pr = P^{-1}, \text{ where } P^5 = 1,$$

$$tP = Pt, \quad sP = Ps,$$

$$(s^{-1}ts)^{-1}P(s^{-1}ts) = s^{-1}t^{-1}(sPs^{-1})ts = s^{-1}t^{-1}Pts = s^{-1}Ps = P$$

whereas $s^{-1}ts = r$ and $r^{-1}Pr = P^{-1}$. This is a contradiction.

So in this case the elements of order 2 correspond to the identical automorphism. Thus there are 2 groups in this case.

So altogether there are $18 + 4 + 12 + 2 = 36$ groups of order 180 with a normal subgroup of order 5, and one with 6 Sylowgroups of order 5. The 37 groups will be represented by generators and relations.

§ 2. INTERRELATION BETWEEN THE TWO METHODS

Let G be any intransitive group of degree $x_1 + x_2$ on two systems of intransitivity, x_1 symbols α 's and x_2 symbols β 's.

Let g_1 be the normal subgroup of G which leaves invariant all the β 's.

$$g_2 \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \alpha's.$$

The totality of different permutations on the α 's alone, each of which is produced by an element of G , forms a permutation group G_1 on the α 's, such that $G_1 \cong G/g_2$. Similarly, if G_2 is the permutationgroup formed by the permutation of G when their effect on the β 's are left out of account, then $G_2 \cong G/g_1$. Also $G/g_1 \times g_2 \cong G_1/g_1 \cong G_2/g_2$.

and the permutations of G are given each once and once only by multiplying each permutation of G_1 with the corresponding ($g_2:1$) permutations of G_2 . If G has more than two systems of intransitivity, the general case is ¹:

if (1) G_r is what G becomes when the permutations of G are performed on the r^{th} set only,

(2) Γ_r is what G becomes when the permutations of G are performed on all the sets except the r^{th} ,

(3) g_r is that subgroup of G which changes the symbols of the r^{th} set only,

(4) γ_r is that subgroup of G which keeps all the symbols of the r^{th} set unchanged,

then $G_r/g_r \cong \Gamma_r/\gamma_r$

and the permutations of G are given each once and once only by multiplying each permutation of G_r by the ($\gamma_r:1$) permutations of Γ_r that correspond to it.

Conversely,² if G and H are two permutation groups on different symbols, such that to each permutation of G there correspond m permutations of H and if to each permutation of H there correspond n permutations of G and if this correspondence satisfies the conditions of homomorphism, then, if each permutation of G is multiplied successively with the corresponding permutations of H , the products will form an intransitive group with G and H as the constituents.

Let the case of only two intransitive systems α 's and β 's be considered. G_1 has effect on the α 's only and is transitive in them, and so also is g_1 . Hence G_1/g_1 has effect on the α 's only. Similarly G_2/g_2 has effect on the β 's only. So every permutation of G_1/g_1 is permutable with every permutation of G_2/g_2 .

Hence in the isomorphism $G_1/g_1 \cong G_2/g_2$

if $a \rightarrow a'$

$b \rightarrow b'$

then $ab \rightarrow a'b'$

and $aa'bb' = aba'b'$.

So if h_1 is a representative of a class of G_1/g_1 ,

and $h_2 \dots \dots G_2/g_2$,

then the set $H = (h_1 h_2) = (h)$ forms a group.

From $G/g_1 \times g_2 \cong G_1/g_1 \cong G_2/g_2 \cong H \dots (1)$

¹ Burnside: *loc. cit.* §142-43-44.

² Netto: *Substitutionentheorie*. § 93.

G can be obtained by Schreier's method, if a homomorphism can be established between H and a subgroup of the group of automorphisms of $g_1 \times g_2$. The number of non-isomorphic G 's will depend on the number of non-isomorphic H 's and different homomorphisms. From the method of its formation H is unique (in the sense of isomorphism). Hence the number of G 's will depend on the number of different homomorphisms.

It will be proved here that there is only one homomorphism between H and a subgroup of the group of automorphisms of $g_1 \times g_2$; so that there is only one G with relations (1).

Suppose $h = h_1 h_2$. Since g_1 and g_2 are not isomorphic, they can not be transformed into one another by any automorphism. So to each h there will correspond an automorphism of g_1 and an automorphism of g_2 ,

$$\text{i.e., } h^{-1}g_1h = g_1, \quad h^{-1}g_2h = g_2.$$

$$\text{But } h^{-1}g_1h = h_2^{-1}h_1^{-1}g_1h_1h_2 = h_2^{-1}g_1h_2.$$

$$\begin{array}{ll} \text{Again} & g_1 \text{ has effect on the } \alpha\text{'s only,} \\ \text{whereas} & \begin{array}{cccc} G_2/g_2 & \dots & \dots & \beta\text{'s} \dots \\ \therefore h_2 & \dots & \dots & \beta\text{'s} \dots \end{array} \end{array}$$

Hence h_2 is permutable with every permutation of g_1 , i.e., h_2 corresponds to the identical automorphism of g_1 and to the automorphism $h_2^{-1}g_2h_2$ of g_2 .

Thus to the elements of H there corresponds only one automorphism of $g_1 \times g_2$.

$$\text{With the notations } h^{-1}gh = g^{\theta_h} \text{ and } G_1 = G_{k_1 x_1}^{x_1}$$

$$G_2 = G_{k_2 x_2}^{x_2}$$

one gets the theorem:

Theorem.

The generating relations of $G = (G_{k_1 x_1}^{x_1} : G_{k_2 x_2}^{x_2})_{k_3}$ are

$$(i) \quad \{g_1, g_2\} = g_1 \times g_2$$

$$(ii) \quad g_1^{\theta_h} = g_1^{\theta_{h_1}}; \quad g_2^{\theta_h} = g_2^{\theta_{h_2}}$$

$$g_1^{\theta_{h_2}} = g_1^{\theta_1}; \quad g_2^{\theta_{h_1}} = g_2^{\theta_1}.$$

where $g_1, g_2, h_1, h_2, \theta_h$ have the same meaning as before, and

G_{kx}^x = the transitive permutation group of degree x and order kx

$(G_{k_1 x_1}^{x_1} : G_{k_2 x_2}^{x_2})_{k_3}$ denotes that the two groups are combined together by isomorphism according to a quotient group of order k_3 .

GROUPS OF ORDER 180

I	II Quotient group of order 36	III Representation by Intransitive groups	IV Generating relations
1	A_3	$(G_4^1 : G_2^2 : G_2^2)_1 (a)$	$P^2 = r^4 = s^2 = 1, rs = sr, sP = Ps, rP = Pr.$
2	"	$((G_4^1 : G_{10}^2)_2 : G_2^2)_1 (c)$	$P^2 = r^4 = s^2 = 1, rs = sr, sP = Ps, r^{-1}Pr = P^{-1}.$
3	"	$((G_{20}^2 : G_4^1) : G_2^2)_1 (e)$	$P^2 = r^4 = s^2 = 1, rs = sr, \quad r^{-1}Pr = P^3.$
4	A_5	$((G_4^1 : G_{10}^2)_2 : G_2^2)_1 (b)$	$\dots \quad r^{-1}sr = s^{-1}, sP = Ps, rP = Pr.$
5	"	$((G_{10}^2 : G_{10}^2)_2 : G_4^1)_2 (d')$	$\dots \quad \dots \quad r^{-1}Pr = P^{-1}$
6	"	$((G_{20}^2 : G_4^1) : G_{10}^2)_2 (f)$	$\dots \quad \dots \quad r^{-1}Pr = P^3.$
7	A_7	(a)	$P^5 = s_1^2 = s_2^2 = r^4 = 1, rs = sr, rP = Pr.$
8	"	(c)	$\dots \quad \dots \quad r^{-1}Pr = P^{-1}.$
9	"	(e)	$\dots \quad \dots \quad r^{-1}Pr = P^3.$
10	A_8	$((G_{20}^2 : G_4^1) : G_2^2)_1 (h)$	$P^5 = s_1^2 = s_2^2 = r^4 = 1, r^{-1}s_1r = s_2, r^{-1}s_2r = s_1, rP = Pr.$
11	"	$((G_{20}^2 : G_4^1) : G_{10}^2)_2 (i)$	$\dots \quad \dots \quad r^{-1}Pr = P^{-1}$
12	"	$((G_{20}^2 : G_4^1) : G_{20}^2)_4 (g)$	$\dots \quad \dots \quad r^{-1}Pr = P^3.$
13	A_{11}	(b)	$P^5 = r^4 = s_1^2 = s_2^2 = 1, s_1s_2 = s_2s_1, s_1P = Ps_1, r^{-1}s_1r = s_1^{-1}, rP = Pr.$
14	"	(d')	$\dots \quad \dots \quad r^{-1}Pr = P^{-1}$
15	"	(f)	$\dots \quad \dots \quad r^{-1}Pr = P^3.$
16	A_{12}	(b)	$P^5 = r^4 = s_1^2 = s_2^2 = 1, s_1s_2 = s_2s_1, s_1P = Ps_1, r^{-1}s_1r = s_1^{-1}, rs_2 = s_2r, rP = Pr.$
17	"	(d')	$\dots \quad \dots \quad r^{-1}Pr = P^{-1}.$
18	"	(f)	$\dots \quad \dots \quad r^{-1}Pr = P^3.$
19	A_4	(a)	$P^2 = s^2 = r^2 = t^2 = 1, rt = tr, rs = sr, sP = Ps, rP = Pr, tP = Pt.$
20	"	(c)	$\dots \quad \dots \quad rPr = P^{-1}, tP = Pt.$

GROUPS OF ORDER 180 (contd.)

I	II Quotient group of order 36	III Representation by Intransitive groups	IV Generating relations
21	A_{10}	(a)	$P^6 = s_1^3 = s_2^3 = r^2 = t^2 = 1$, $rs_1 = s_1r$, $ts_1 = s_1t$, $s_1s_2 = s_2s_1$, $rt = tr$, $rP = Pr$, $tP = Pt$
22	"	(c)	 $rPr = P^{-1}$, $tP = Pt$.
23	A_6	(b)	$P^6 = s_1^3 = r^2 = t^2 = 1$, $rt = tr$, $tst = s^{-1}$, $rs = sr$, $rP = Pr$, $tP = Pt$.
24	"	$(G_{12}^2 : G_{10}^6)_1 : G_4^2 (d'')$	 $rPr = P^{-1}$, $tP = Pt$.
25	"	(d')	 $rP = Pr$, $tPt = P^{-1}$.
26	A_6	(h)	$P^6 = s_1^3 = s_2^3 = r^2 = t^2 = 1$, $rs_1r = s_1^{-1}$, $ts_1t = s_1^{-1}$, $ts_2 = s_2t$, $tP = Pt$, $rP = Pr$.
27	"	(i)	 $tP = Pt$, $rPr = P^{-1}$.
28	"	(i)	 $tPt = P^{-1}$, $rP = Pr$.
29	A_{12}	(b)	$P^6 = s_1^3 = s_2^3 = r^2 = t^2 = 1$, $rs_1r = s_1^{-1}$, $ts_1t = s_1^{-1}$, $rP = Pr$, $tP = Pt$.
30	"	(d')	 $rPr = P^{-1}$, $tP = Pt$.
31	"	(d'')	 $rP = Pr$, $tPt = P^{-1}$.
32	A_{12}	(b)	$P^6 = s_1^3 = s_2^3 = r^2 = t^2 = 1$, $rs_1r = s_1^{-1}$, $rs_2 = s_2r$, $ts_1 = s_1t$, $rP = Pr$, $tP = Pt$.
33	"	(d')	 $rPr = P^{-1}$, $tP = Pt$.
34	"	(d'')	 $rP = Pr$, $tPt = P^{-1}$.
35	A_1	$[(G_{12}^4 : G_6^2)_3 : G_3^2]_1 (j)$	$P^6 = s^9 = r^2 = t^2 = 1$, $rt = tr$, $sP = Ps$, $s^{-1}ts = r$, $s^{-1}rs = tr$, $rP = Pr$, $tP = Pt$.
36	A_2	(j)	$P^6 = s_1^3 = s_2^3 = r^2 = t^2 = 1$, $rt = tr$, $s_1s_2 = s_2s_1$, $s_1P = Ps_1$, $s_2t = ts_2$, $s_2r = rs_2$, $s_1^{-1}ts_1 = r$, $s_1^{-1}rs_1 = tr$.
37	$\times$	$\times$	$P^6 = s_1^3 = s_2^3 = 1$, $(s_1P)^2 = 1$, $s_2P = Ps_2$, $s_1s_2 = s_2s_1$. Direct product of the icosahedral group and $s_2^3 = 1$.

The notations (a), (b), ..., (h) are taken from Senior and Lunn's paper, American Journal of Math., Vol. 56.

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ON THE D^2 -STATISTIC

By

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The insistent need of having a suitable measure of the divergence between two or more multivariate normal populations, once a significant difference between these has been established, led Professor P. C. Mahalanobis² in 1927 to develop what has now come to be regarded as the "Mahalanobis' Generalised Distance" or the D^2 -statistic. Mere tests of significance do not completely bring to light the exact nature of the difference between the two populations, unless an estimate of the magnitude of that difference is, in some way, obtainable, independently of the sample size. This statistical object has been very aptly achieved by this statistic.

The moments of the D^2 -statistic were first worked out by the author of the measure himself¹ who also laid stress later² on the possibilities of its wider generalisation. On the assumption of the knowledge of the system of variances and covariances of the two populations concerned in the construction of the D^2 -statistic (the classical D^2), its sampling distribution was first worked out by R. C. Bose³ and critical study of this distribution law was then made by S. S. Bose.⁶ Later on R. C. Bose⁴ also showed that the mean of independent samples, drawn from the D^2 -population, also followed the same general law.

When, however, the system of variances and covariances are not known, they have to be estimated from the samples. Consequently, for tests of significance, an exact distribution of studentized D^2 is needed. This was worked out by R. C. Bose and S. N. Ray⁵ in 1938.

In this paper the distribution laws regarding the classical D^2 -statistic have been obtained by the method of characteristic functions, and two new distribution laws have been obtained—those for the difference and quotient of two independent values of D^2 . It has been shown elsewhere⁷ that the moments of D^2 , obtained previously by a

rather long process, can easily be obtained by expanding the Logarithm of the corresponding characteristic function.

1. Definition of the Characteristic Function

If $f(x_1, x_2, \dots, x_p)$ represent the frequency function of the variates $x_1, x_2, \dots, x_p$, such that

$$\iint \dots \int_R f(x_1, x_2, x_3, \dots, x_p) dx_1 dx_2 \dots dx_p = 1 \quad \dots (1)$$

where R is the region of applicability of $f(x_1, x_2, \dots, x_p)$, then the characteristic function (c.f.) of the system $x_1, x_2, \dots, x_p$ is given by

$$\phi(t_1, t_2, \dots, t_p) = \iiint \dots \int_R \exp\left\{i \sum_{\mu=1}^p t_{\mu} x_{\mu}\right\} \cdot f(x_1, \dots, x_p) dx_1 \dots dx_p \quad (2)$$

where t 's are real quantities lying in the suitable range, generally $(-\infty, \infty)$. We may also write this in the form

$$\phi(t_1, t_2, \dots, t_p) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp\left\{i \sum_{\mu=1}^p t_{\mu} x_{\mu}\right\} \cdot f(x_1, \dots, x_p) dx_1 \dots dx_p,$$

provided we agree that $f(x_1, x_2, \dots, x_p) \equiv 0$ outside the region R .

The relation between f and ϕ is given by the Fourier-transform,

$$f(x_1, x_2, \dots, x_p) = \left(\frac{1}{2\pi}\right)^p \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp\left\{-i \sum_{\mu=1}^p t_{\mu} x_{\mu}\right\} \cdot \phi(t_1, \dots, t_p) dt_1 \dots dt_p. \quad (3)$$

Again, if $U_{\mu} \equiv U_{\mu}(x_1, x_2, \dots, x_p)$, ($\mu=1, 2, \dots, r$) be functions of the variables $x_1, \dots, x_p$ whose distribution law (d.l.) is $f(x_1, x_2, \dots, x_p)$, then the c.f. of the d.l. of U_{μ} 's is given by

$$\phi(t_1, t_2, \dots, t_r) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp\left\{i \sum_{\mu=1}^r t_{\mu} U_{\mu}\right\} \cdot f(x_1, x_2, \dots, x_p) dx_1 \dots dx_p. \quad (4)$$

and the d.l. of the U 's is given by

$$P(U_1, U_2, \dots, U_r) = \left(\frac{1}{2\pi}\right)^r \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp\left\{-i \sum_{\mu=1}^r t_{\mu} U_{\mu}\right\} \cdot \phi(t_1, \dots, t_r) dt_1 \dots dt_r. \quad (5)$$

2. The multivariate Normal Population

The normal population in p correlated variates is given by

$$f = (2\pi)^{-\frac{1}{2}p} |a_{\mu\nu}|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2} \sum_{\mu=1}^p \sum_{\nu=1}^p a^{\mu\nu} (x_{\mu} - m_{\mu})(x_{\nu} - m_{\nu})\right\} \quad (6)$$

where m_μ is the population mean of the p^{th} variate, $\alpha_{\mu\nu} = \sigma_\mu \sigma_\nu \rho_{\mu\nu}$ where σ_μ is the standard deviation of the μ^{th} character, $\rho_{\mu\nu}$ is the coefficient of correlation between the μ^{th} and the ν^{th} character and

$$\alpha^{\mu\nu} = \frac{\text{cofactor of } \alpha_{\mu\nu} \text{ in } |\alpha_{\mu\nu}|}{|\alpha_{\mu\nu}|}.$$

If we write $\beta_{\mu\nu}$ for $\alpha^{\mu\nu}$, and denote

$$\sum_{\mu=1}^p \sum_{\nu=1}^p \beta_{\mu\nu} (x_\mu - m_\mu)(x_\nu - m_\nu)$$

by $B(x-m, x-m)$, where $B \equiv \|\beta_{\mu\nu}\|$ is a matrix of the positive definite quadratic form to which this expression refers, then (6) can be written as

$$f = (2\pi)^{-\frac{1}{2}p} |\beta_{\mu\nu}|^{-\frac{1}{2}} \exp\{-\frac{1}{2}B(x-m, x-m)\}. \quad \dots \quad (7)$$

With respect to (7) three standard results* have to be noted. That

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp\{-\frac{1}{2}B(x-m, x-m)\} \prod_{\mu=1}^p dx_\mu = (2\pi)^{\frac{1}{2}p} |\beta_{\mu\nu}|^{-\frac{1}{2}} \quad \dots \quad (8)$$

is very obvious.

Next we find that

$$\begin{aligned} & B(x-m \pm t', x-m \pm t') \\ &= \sum_{\mu=1}^p \sum_{\nu=1}^p \beta_{\mu\nu} (x_\mu - m_\mu \pm t'_\mu)(x_\nu - m_\nu \pm t'_\nu) \\ &= \sum_{\mu=1}^p \sum_{\nu=1}^p \beta_{\mu\nu} (x_\mu - m_\mu)(x_\nu - m_\nu) \\ &\quad + 2 \sum_{\mu=1}^p \sum_{\nu=1}^p (x_\mu - m_\mu) \beta_{\mu\nu} t'_\nu + B(t', t') \\ &= B(x-m, x-m) \pm 2 \sum_{\mu=1}^p t'_\mu (x_\mu - m_\mu) + B^{-1}(t, t), \end{aligned}$$

where

$$t'_\mu = \sum_{\nu=1}^p \beta_{\mu\nu} t'_\nu, \text{ and so } t'_\mu = \sum_{\nu=1}^p \beta^{\mu\nu} t_\nu,$$

and

$$B^{-1}(t, t) = \sum_{\mu=1}^p \sum_{\nu=1}^p \beta^{\mu\nu} t_{\mu} t_{\nu} = B(t', t');$$

i.e., $B^{-1}(t, t)$ is the reciprocal quadratic form. Hence we have an important relation

$$\begin{aligned} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}B(x-m, x-m) \pm i \sum_{\mu=1}^p t_{\mu} (x_{\mu} - m_{\mu})\right\} \prod_{\mu=1}^p dx_{\mu} \\ = (2\pi)^{\frac{1}{2}p} |\beta_{\mu\nu}|^{-\frac{1}{2}} \cdot \exp\left\{-\frac{1}{2}B^{-1}(t, t)\right\}. \quad \dots (9) \end{aligned}$$

Again, if a random sample of size n be drawn from (7), and $\bar{x}_{\mu}$ denote the sample mean of the μ^{th} character, $\mu=1, 2, \dots, p$, then the d.l. of $\bar{x}$ is given by

$$\bar{f} = (2\pi)^{-\frac{1}{2}p} |n\beta_{\mu\nu}|^{\frac{1}{2}} \cdot \exp\left\{-\frac{1}{2}nB(\bar{x}-m, \bar{x}-m)\right\}. \quad \dots (10)$$

3. Distribution of the Mean Difference

If we have another normal population of p correlated variates $x'_1, x'_2, \dots, x'_p$ and if $\bar{x}'_{\mu}$ denote the mean of the μ^{th} character x'_{μ} for a random sample of size n' then, the d.l. of $\bar{x}'$ is, like (10), given by

$$\bar{f}' = (2\pi)^{-\frac{1}{2}p} |n'\beta'_{\mu\nu}|^{\frac{1}{2}} \cdot \exp\left\{-\frac{1}{2}n'B'(\bar{x}'-m', \bar{x}'-m')\right\}. \quad \dots (11)$$

From (10) and (11) we find that the c.f. of the d.l. of the differences $\bar{x}_{\mu} - \bar{x}'_{\mu}$ ($\mu=1, 2, \dots, p$) is given by

$$\begin{aligned} \phi_1(t_1, t_2, \dots, t_p) &= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{f} \bar{f}' \prod_{\mu=1}^p dx_{\mu} dx'_{\mu} \cdot \exp\left\{i \sum_{\mu=1}^p t_{\mu} (\bar{x}_{\mu} - \bar{x}'_{\mu})\right\} \\ &= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{f} \bar{f}' \exp\left[i \sum_{\mu=1}^p t_{\mu} \{(\bar{x}_{\mu} - m_{\mu}) - (\bar{x}'_{\mu} - m'_{\mu}) + \right. \\ &\quad \left. (m'_{\mu} - m_{\mu})\}\right] \prod_{\mu=1}^p dx_{\mu} dx'_{\mu} \\ &= \exp\left[i \sum_{\mu=1}^p t_{\mu} (m'_{\mu} - m_{\mu}) - \frac{1}{2}\{(nB)^{-1}(t, t) + \right. \\ &\quad \left. (n'B')^{-1}(t, t)\}\right], \quad \dots (12) \end{aligned}$$

in virtue of (9), where $(nB)^{-1}$ and $(n'B')^{-1}$ are the reciprocal matrices of $\|n\beta_{\mu\nu}\|$ and $\|n'\beta'_{\mu\nu}\|$ respectively.

Combining the two matrices in (12) we can write it as

$$\phi_1(t_1, t_2, \dots, t_p) = \exp\left\{i \sum_{\mu=1}^p t_{\mu} (m_{\mu} - m'_{\mu}) - \frac{1}{2} C(t, t)\right\}, \quad \dots (13)$$

where it can easily be verified that C stands for the matrix $\|\gamma_{\mu\nu}\|$

where $\gamma_{\mu\nu} = \frac{a_{\mu\nu}}{n} + \frac{a'_{\mu\nu}}{n'}$, with the corresponding determinant

$$|\gamma_{\mu\nu}| = \left| \frac{a_{\mu\nu}}{n} + \frac{a'_{\mu\nu}}{n'} \right|.$$

Applying the inversion formula, we find that the d.l. of the differences $\bar{x}_{\mu} - \bar{x}'_{\mu}$ ($\mu=1, 2, \dots, p$) is given by

$$\begin{aligned} \bar{f} &= (2\pi)^{-\frac{1}{2}p} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \phi_1(t_1, t_2, \dots, t_p) \\ &\quad \times \exp\left\{-i \sum_{\mu=1}^p t_{\mu} (\bar{x}_{\mu} - \bar{x}'_{\mu})\right\} \cdot \prod_{\mu=1}^p dt_{\mu} \\ &= (2\pi)^{-\frac{1}{2}p} |\gamma_{\mu\nu}|^{-\frac{1}{2}} \cdot \exp\left\{-\frac{1}{2} C^{-1}(X-M, X-M)\right\}, \quad \dots (14) \end{aligned}$$

where we write $\bar{X}_{\mu}$ for $\bar{x}_{\mu} - \bar{x}'_{\mu}$ and $\bar{M}_{\mu}$ for $M_{\mu} - M'_{\mu}$ and where C^{-1} again stands for the reciprocal matrix of C.

In the special case, in which we are more interested here, namely when the system of variances and covariances of the two populations given by (10) and (11) are the same, i.e.,

$$a_{\mu\nu} = a'_{\mu\nu},$$

(14) reduces to

$$\bar{f}_1 = (2\pi)^{-\frac{1}{2}p} \left| \frac{n}{2} \beta_{\mu\nu} \right|^{\frac{1}{2}} \cdot \exp\left\{-\frac{n}{4} B(X-M, X-M)\right\}, \quad \dots (15)$$

where

$$\frac{n}{2} = \frac{1}{n} + \frac{1}{n'}.$$

4. *Distribution of the Classical D²-Statistic*

When the two multivariate normal populations, each having the same number, p , of correlated variates, say $x_1, x_2, \dots, x_p$, and $x'_1, x'_2, \dots, x'_p$, have the same set of variances and covariances, namely $\alpha_{\mu\nu}$ but different means m_μ and m'_μ ($\mu=1, 2, \dots, p$) then D^2 is defined by the relation

$$D^2 = D_1^2 - 2/\bar{n}, \quad \dots (17)$$

$$\text{where} \quad D_1^2 = \frac{1}{p} \sum_{\mu=1}^p \sum_{\nu=1}^p \alpha^{\mu\nu} X_\mu X_\nu = \frac{1}{p} B(X, X), \quad \dots (18)$$

the symbols involved being defined as in the above sections.

Likewise, the generalised distance between the population means is given by

$$\Delta^2 = \frac{1}{p} B(M, M) \quad \dots (19)$$

From (15) and (18) we find that the c.f. of the d.l. of pD_1^2 is given by

$$\begin{aligned} \phi_2(t) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \bar{f}_1 \cdot \exp(itpD_1^2) \cdot \prod_{\mu=1}^p dX_\mu, \\ &= (2\pi)^{-\frac{1}{2}p} \left| \frac{\bar{n}}{2} \beta_{\mu\nu} \right|^{\frac{1}{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp \left[-\frac{\bar{n}}{2} \left\{ B(X, X) - 2 \sum_{\mu=1}^p X_\mu M'_\mu \right. \right. \\ &\quad \left. \left. + B(M, M) \right\} + itB(X, X) \right] \prod_{\mu=1}^p dX_\mu, \quad \dots (20) \end{aligned}$$

$$\text{where} \quad M'_\mu = \sum_{\nu=1}^p \frac{\bar{n}}{2} \beta_{\mu\nu} M_{\mu\nu}.$$

If we write $G = \|g_{\mu\nu}\|$, where $g_{\mu\nu} = \left(\frac{\bar{n}}{2} - it \right) \beta_{\mu\nu}$,

(20) reduces to

$$\begin{aligned}
\phi_2(t) &= (2\pi)^{-\frac{1}{2}p} \left| \frac{\bar{n}}{2} \beta_{\mu\nu} \right|^{\frac{1}{2}} \cdot \exp \left(-\frac{\bar{n}}{4} p \Delta^2 \right) \\
&\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp \left\{ -\frac{1}{2} G(X, X) + \sum_{\mu=1}^p \left(\frac{\bar{n}}{2} M'_{\mu} \right) X_{\mu} \right\} \prod_{\mu=1}^p dx_{\mu} \\
&= \left| \frac{\bar{n}}{2} \beta_{\mu\nu} \right|^{\frac{1}{2}} \left| g_{\mu\nu} \right|^{\frac{1}{2}} \exp \left\{ -\frac{\bar{n}}{4} p \Delta^2 + \frac{1}{2} G^{-1} \left(\frac{\bar{n}}{2} M', \frac{\bar{n}}{2} M' \right) \right\} \\
&= \left(\frac{\bar{n}}{\bar{n} - 4it} \right)^{\frac{1}{2}p} \cdot \exp \left\{ -\frac{\bar{n}}{4} p \Delta^2 + \frac{\bar{n}^2}{8} \cdot \frac{2}{\bar{n} - 4it} B^{-1}(M', M') \right\} \\
&= \left(\frac{\bar{n}}{\bar{n} - 4it} \right)^{\frac{1}{2}p} \cdot \exp \left\{ -\frac{\bar{n}}{4} p \Delta^2 + \frac{\bar{n}^2 p \Delta^2}{4(\bar{n} - 4it)} \right\}. \quad \dots (21)
\end{aligned}$$

The distribution law of pD_1^2 is therefore given by

$$\begin{aligned}
P(pD_1^2) &= (\bar{n})^{\frac{1}{2}p} e^{-\frac{\bar{n}}{4} p \Delta^2} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} (\bar{n} - 4it)^{-\frac{1}{2}p} \\
&\quad \times \exp \left\{ -it p D_1^2 + \frac{\bar{n}^2 p \Delta^2}{4(\bar{n} - 4it)} \right\} dt. \quad \dots (22)
\end{aligned}$$

Putting $pD_1^2(\bar{n} - 4it)/4 = T$,
this becomes

$$\begin{aligned}
P(pD_1^2) &= \frac{1}{4} (\bar{n})^{\frac{1}{2}p} \cdot e^{-\frac{\bar{n}}{4} p(\Delta^2 + D_1^2)} \cdot \left(\frac{pD_1^2}{4} \right)^{\frac{1}{2}(p-2)} \\
&\quad \times \frac{1}{2\pi i} \int_{\frac{pD_1^2}{4} - i\infty}^{\frac{pD_1^2}{4} + i\infty} \frac{\exp \left\{ T + \frac{\bar{n}^2 p^2 \Delta^2 D_1^2}{4T} \right\}}{T^{\frac{1}{2}p}} dT. \quad \dots (23)
\end{aligned}$$

Since the integrand in (23) is regular throughout the T -plane except at the points 0 and ∞ , we can, by Cauchy's theorem, change the path of integration from that given above to the one which starts from $-i\infty$, encircles the origin once in the positive direction, and finally returns to $-i\infty$. Hence we can write (23) in the form,

$$P(pD_1^2) = H_1 \cdot \frac{1}{2\pi i} \int_{-i\infty}^{(0+)} T^{-\frac{1}{2}p} \cdot \exp \left\{ T + \frac{\bar{n}^2 p^2 D_1^2 \Delta^2}{T} \right\} dT, \quad \dots (24)$$

where $H_1 = \frac{1}{4}(\bar{n})^{\frac{1}{2}} p (pD_1^2/4)^{\frac{1}{2}(p-2)} \exp \left\{ -\frac{\bar{n}}{4} p (D_1^2 + \Delta^2) \right\}$.

But we know that

$$\frac{1}{2\pi i} \int_{-\infty i}^{(0+)} t^{-n-1} \exp \left\{ t + \frac{z^2}{4t} \right\} dt = \left(\frac{z}{2} \right)^{-n} I_n(z). \quad \dots (25)$$

Hence the d.l. of D_1^2 is given by

$$P(D_1^2) = \frac{\bar{n}p}{4} (D_1^2 / \Delta^2)^{\frac{1}{2}(p-2)} e^{-\frac{\bar{n}}{4} p (D_1^2 + \Delta^2)} I_{\frac{1}{2}(p-2)} \left(\frac{\bar{n}p}{2} \sqrt{\Delta^2 D_1^2} \right). \quad \dots (26)$$

Finally, putting $D_1^2 = D^2 + \frac{2}{n}$, we get the d.l. of D^2 as

$$P(D^2) = \frac{\bar{n}p}{4} \left(\frac{\bar{n}D^2 + 2}{\bar{n}\Delta^2} \right)^{\frac{1}{2}(p-2)} e^{-\frac{\bar{n}p}{4} \left(D^2 + \Delta^2 + \frac{2}{n} \right)} \times I_{\frac{1}{2}(p-2)} \left\{ \frac{\bar{n}p}{2} \sqrt{\Delta^2 \left(D^2 + \frac{2}{n} \right)} \right\} \quad \dots (27)$$

as given by R. C. Bose.³

In the special case, when $\Delta^2 = 0$, the c.f. of the d.l. of pD_1^2 reduces to

$$\phi'_2(t) = \{ \bar{n} / (\bar{n} - 4it) \}^{\frac{1}{2}p}, \quad \dots (28)$$

so that in this case the d.l. of pD_1^2 is given by

$$P'(pD_1^2) = \frac{1}{2\pi} \int_0^\infty \left(\frac{\bar{n}}{\bar{n} - 4it} \right)^{\frac{1}{2}p} e^{-itpD_1^2} dt,$$

which, after putting $\frac{1}{4}(\bar{n} - 4it)pD_1^2 = -T$ and suitably changing the path of integration (which is permissible), becomes

$$\begin{aligned} P'(pD_1^2) &= \frac{1}{4}(\bar{n})^{\frac{1}{2}} p (pD_1^2/4)^{\frac{1}{2}p-1} e^{-\bar{n}pD_1^2/4} \cdot \frac{i}{2\pi} \int_{\infty}^{(0+)} (-T)^{-\frac{1}{2}p} e^{-T} dT \\ &= \frac{1}{4}(\bar{n})^{\frac{1}{2}} p (pD_1^2/4)^{\frac{1}{2}p-1} e^{-\bar{n}pD_1^2/4} / \Gamma(\tfrac{1}{2}p). \quad \dots (29) \end{aligned}$$

Hence the *d.l.* of D_1^2 is given by

$$P(D_1^2) = \frac{1}{\Gamma(\frac{1}{2}p)} \left(\frac{\bar{n}p}{4} \right)^{\frac{1}{2}p} e^{-\bar{n}p D_1^2/4} (D_1^2)^{\frac{1}{2}(p-2)} \quad \dots \quad (30)$$

and that of D^2 by

$$P(D^2) = \frac{1}{\Gamma(\frac{1}{2}p)} \left(\frac{\bar{n}p}{4} \right)^{\frac{1}{2}p} e^{-\frac{\bar{n}p}{4} \left(D^2 + \frac{2}{n} \right)} \left(D^2 + \frac{2}{n} \right)^{\frac{1}{2}(p-2)} \quad \dots \quad (31)$$

5. The Distribution Law of the Mean $\bar{D}^2$

Let there be k pairs of independent samples drawn at random from the same pair of p -variate normal populations, as used above, giving us the k values of D_1^2 , viz.,

$$D_{11}^2, D_{12}^2, \dots, D_{1k}^2;$$

D_{1k}^2 standing for the value of D_1^2 from the k^{th} sample.

Then it is easily seen from equation (21) that the *c.f.* of the *d.l.* of the mean $\bar{D}_1^2$,

$$\text{where} \quad \bar{D}_1^2 = \frac{1}{k} \sum_{\mu=1}^k D_{1\mu}^2,$$

is given by

$$\phi_2(t) = \left(\frac{\bar{n}pk}{\bar{n}pk - 4it} \right)^{\frac{1}{2}pk} \exp \left\{ -\frac{\bar{n}kp\Delta^2}{2} + \frac{k^2\bar{n}^2p^2\Delta^2}{4(\bar{n}pk - 4it)} \right\} \quad \dots \quad (32)$$

From the forms of (32) and (21), we find at once that the *d.l.* of $\bar{D}_1^2$ is given by

$$P(\bar{D}_1^2) = \frac{\bar{n}pk}{4} \left(\frac{\bar{D}_1^2}{\Delta^2} \right)^{\frac{pk-2}{4}} e^{-\bar{n}pk(\bar{D}_1^2 + \Delta^2)/4} \times I_{\frac{1}{2}(pk-2)} \left(\frac{\bar{n}pk}{2} \sqrt{\Delta^2 \bar{D}_1^2} \right) \quad \dots \quad (33)$$

and that of $\overline{D^2} = \overline{D_1^2} - \frac{2}{n}$, by

$$P(\overline{D^2}) = \frac{\bar{n}pk}{4} \left(\frac{\bar{n}^2 \overline{D^2} + 2}{\bar{n} \Delta^2} \right)^{\frac{1}{4}(pk-2)} \cdot \exp \left\{ -\bar{n}pk \left(\overline{D^2} + \Delta^2 + \frac{2}{n} \right) / 4 \right\} \\ \times I_{\frac{1}{2}(pk-2)} \left\{ \frac{\bar{n}pk}{2} \sqrt{\Delta^2 (\overline{D^2} + 2/n)} \right\} \dots \quad (34)$$

Similarly, it can easily be seen that when $\Delta^2 = 0$, the d.l. of $\overline{D^2}$ is given by

$$P(\overline{D^2}) = \frac{1}{\Gamma\left(\frac{pk}{2}\right)} \left(\frac{\bar{n}pk}{4} \right)^{\frac{1}{2}pk} e^{-\frac{\bar{n}pk}{4} \left(\overline{D^2} + \frac{2}{n} \right)} \left(\overline{D^2} + \frac{2}{n} \right)^{\frac{1}{2}(pk-2)} \dots \quad (35)$$

6. Change of notation

For convenient handling of the mathematical processes, we shall from hereonward use transformed forms of the distribution laws given in (26) and (30).

If we set

$$m = \frac{1}{2}(p-2), \quad \alpha = \frac{\bar{n}p}{4}, \quad q = 2\alpha\Delta, \quad \text{and } D_1^2 = x, \quad \dots \quad (36)$$

the equations (26) and (30) transform into

$$P(x) = H_2 e^{-\alpha x} x^{\frac{1}{2}m} I_m(q\sqrt{x}), \quad \dots \quad (37)$$

$$\text{where} \quad H_2 = (2/q)^m \alpha^{m+1} e^{-q^2/4\alpha}, \quad \dots \quad (38)$$

$$\text{and} \quad P(x) = \frac{\alpha^{m+1}}{\Gamma(m+1)} e^{-\alpha x} x^m, \quad \dots \quad (39)$$

respectively.

Again, it may be noted here that, although we have obtained (31) independently of (26), it could have been derived from it by making q tend to 0, remembering that

$$\lim_{z \rightarrow 0} \frac{2^k I_k(z)}{z^k} = \frac{1}{\Gamma(k+1)}. \quad \dots \quad (40)$$

In deducing results for (39) from the corresponding expressions for (37), we shall now make use of the relation (40).

7. The Distribution Law of the Difference

Suppose now that we have two independent observations, x_1 and x_2 , made on the population represented by (37): to find the *d.l.* of their difference,

$$u = x_1 - x_2, \quad u > 0 \quad \dots (41)$$

Then it is easy to see that the *c.f.* of the *d.l.* of u is given by

$$\begin{aligned} \phi_3(t) &= H_2^2 \int_0^\infty \int_0^\infty e^{it(x_1 - x_2)} e^{-a(x_1 + x_2)} (x_1 x_2)^{\frac{1}{2}m} \\ &\quad \times I_m(q\sqrt{x_1}) I_m(q\sqrt{x_2}) dx_1 dx_2 \\ &= \left(\frac{a^2}{a^2 + t^2} \right)^{m+1} \cdot \exp \left\{ -\frac{q^2}{2a} + \frac{q^2}{2} \cdot \frac{a}{a^2 + t^2} \right\} \dots (42) \end{aligned}$$

Hence by the inversion formula, the *d.l.* of u is given by

$$\begin{aligned} P(u) &= \frac{1}{2\pi} e^{-q^2/2a} \int_{-\infty}^\infty e^{-itu + \frac{q^2}{2} \frac{a}{a^2 + t^2}} \left(\frac{a^2}{a^2 + t^2} \right)^{m+1} dt \\ &= \frac{a^{2m+2}}{2\pi} e^{-q^2/2a} \int_{-\infty}^\infty \sum_{r=0}^\infty \left(\frac{q^2 a}{2} \right)^r \frac{1}{r!} \frac{e^{-itu}}{(a^2 + t^2)^{m+r+1}} dt \\ &= \frac{a^{2m+2}}{2\pi} e^{-q^2/2a} \sum_{r=0}^\infty \left(\frac{q^2 a}{2} \right)^r \frac{1}{r!} \\ &\quad \times \int_{-\infty}^\infty (a^2 + t^2)^{-(m+r+1)} e^{itu} dt. \end{aligned}$$

The change of the order of integration and summation being easily justified. Hence⁹.

$$\begin{aligned} P(u) &= \frac{a^{2m+2}}{2\pi} e^{-q^2/2a} \sum_{r=0}^\infty \left(\frac{q^2 a}{2} \right)^r \frac{1}{r!} \\ &\quad \times \frac{2\sqrt{\pi} u^{m+r+\frac{1}{2}} K_{m+r+\frac{1}{2}}(au)}{\Gamma(m+r+1)(2a)^{m+r+\frac{1}{2}}} \end{aligned}$$

$$= \frac{1}{\sqrt{\pi}} a^{m+\frac{3}{2}} e^{-\frac{q^2}{2a}} \sum_{r=0}^{\infty} \frac{q^{2r} u^{m+r+\frac{1}{2}} K_{m+r+\frac{1}{2}}(au)}{2^{m+2r+\frac{1}{2}} \Gamma(m+r+1) r!} \quad \dots \quad (43)$$

For the population represented by (39), the *d.l.* corresponding to (43) is obtained by putting $q=0$ in (43) and is

$$P(u) = \frac{a^{m+\frac{3}{2}} u^{m+\frac{1}{2}} K_{m+\frac{1}{2}}(au)}{\sqrt{\pi} 2^{m+\frac{1}{2}} \Gamma(m+1)} \quad \dots \quad (44)$$

8. The Distribution of the Ratio

Let x_1 and x_2 be obtained as in the previous section, and let

$$v = x_1/x_2. \quad \dots \quad (45)$$

Put $w = \log v = \log x_1 - \log x_2. \quad \dots \quad (46)$

Then the *c.f.* of the *d.l.* of w is given by

$$\begin{aligned} \phi_4(t) &= \int_0^\infty \int_0^\infty e^{iwt} P(x_1) P(x_2) dx_1 dx_2 \\ &= \int_0^\infty \int_0^\infty (x_1/x_2)^{it} P(x_1) P(x_2) dx_1 dx_2. \quad \dots \quad (47) \end{aligned}$$

From the known result, we see that

$$\begin{aligned} \int_0^\infty e^{-ax} x^{\frac{1}{2}r} I_m(q\sqrt{x}) dx &= \frac{\Gamma\left(\frac{m+r+2}{2}\right) e^{-\frac{q^2}{4a}}}{\Gamma(m+1) a^{\frac{1}{2}(r-1)}} \\ &\times {}_1F_1\left(\frac{m+r+2}{2}; m+1; \frac{q^2}{4a}\right). \quad \dots \quad (48) \end{aligned}$$

Hence, after integration, (47) reduces to

$$\begin{aligned} \phi_4(t) &= \frac{\Gamma(m+1+it)\Gamma(m+1-it)}{\Gamma(m+1)\Gamma(m+1)} e^{-\frac{q^2}{2a}} {}_1F_1\left(m+it+1; m+1; \frac{q^2}{4a}\right) \\ &\times {}_1F_1\left(m+1-it; m+1; \frac{q^2}{4a}\right). \quad \dots \quad (49) \end{aligned}$$

Putting $m+1=\lambda$, $q^2/4a=\theta$, (49) can be written as

$$\phi_4(t) = \frac{\Gamma(\lambda+it)\Gamma(\lambda-it)}{\Gamma(\lambda)\Gamma(\lambda)} e^{-2\theta} {}_1F_1(\lambda+it; \lambda; \theta) {}_1F_1(\lambda-it; \lambda; \theta). \quad (50)$$

Hence the d.l. of w is given by

$$P(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iwt} \phi_4(t) dt. \quad \dots (51)$$

Now it can be shown (10) that

$$\begin{aligned} & {}_1F_1(a; c; x) {}_1F_1(b; c; x) \\ &= \sum_{r=0}^{\infty} \frac{\Gamma(a+r)\Gamma(b+r)\{\Gamma(c)\}^2 x^{2r}}{\Gamma(a)\Gamma(b)\Gamma(c+r)\Gamma(c+2r)r!} {}_1F_1(u+b+2r; c+2r; x). \end{aligned}$$

Hence,

$$\begin{aligned} & {}_1F_1(\lambda+it; \lambda; \theta) {}_1F_1(\lambda-it; \lambda; \theta) \\ &= \sum_{r=0}^{\infty} \frac{\Gamma(\lambda+it+r)\Gamma(\lambda-it+r)\{\Gamma(\lambda)\}^2 \theta^{2r}}{\Gamma(\lambda+it)\Gamma(\lambda-it)\Gamma(\lambda+r)\Gamma(\lambda+2r)r!} {}_1F_1(2\lambda+2r; \lambda+2r; \theta), \end{aligned}$$

and (51) reduces to

$$\begin{aligned} P(w) &= e^{-2\theta} \sum_{r=0}^{\infty} \frac{\theta^{2r} {}_1F_1(2\lambda+2r; \lambda+2r; \theta)}{\Gamma(\lambda+r)\Gamma(\lambda+2r)r!} e^{-i(\lambda+r)w} \\ &\quad \times \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(\lambda+r-it)w} \Gamma(\lambda+it+r) \Gamma(\lambda-it+r) dt, \end{aligned}$$

or putting $\lambda+r-it=-z$, to

$$\begin{aligned} P(w) &= e^{-2\theta} \sum_{r=0}^{\infty} \frac{\theta^{2r} {}_1F_1(2\lambda+2r; \lambda+2r; \theta)}{\Gamma(\lambda+r)\Gamma(\lambda+2r)r!} e^{-(\lambda+r)w} \\ &\quad \times \frac{1}{2\pi i} \int_{-\lambda-r-i\infty}^{-\lambda-r+i\infty} e^{-zw} \Gamma(2\lambda+2r+z) \Gamma(-z) dz. \quad \dots (52) \end{aligned}$$

But we can show ¹¹ that

$$\frac{1}{2\pi i} \int_{-p-i\infty}^{-p+i\infty} e^{-zw} \Gamma(-z) \Gamma(2p+z) dz = \Gamma(2p) (1+e^{-w})^{-2p}. \quad \dots (53)$$

Hence, we find finally that the d.l. of w is

$$\begin{aligned}
 P(w) &= e^{-2\theta} \sum_{r=0}^{\infty} \frac{\theta^{2r} {}_1F_1(2\lambda+2r; \lambda+2r; \theta) \Gamma(2\lambda+2r) e^{-(\lambda+r)w}}{\Gamma(\lambda+r) \Gamma(\lambda+2r) r! (1+e-w)^{2\lambda+2r}} \\
 &= \frac{e^{-2\theta}}{\sqrt{\pi}} \sum_{r=0}^{\infty} \frac{\theta^{2r} 2^{2\lambda+2r-1} \Gamma(\lambda+r+\frac{1}{2}) {}_1F_1(2\lambda+2r; \lambda+2r; \theta) e^{(\lambda+r)w}}{\Gamma(\lambda+2r) r! (1+e^w)^{2\lambda+2r}} \dots (54)
 \end{aligned}$$

Finally, putting $e^w = v$, we have for the distribution law of v

$$\begin{aligned}
 P(v) &= \frac{e^{-2\theta}}{\sqrt{\pi}} \sum_{r=0}^{\infty} \frac{2^{2\lambda-1} (2\theta)^{2r} \Gamma(\lambda+r+\frac{1}{2}) {}_1F_1(2\lambda+2r; \lambda+2r; \theta) v^{\lambda+r-1}}{\Gamma(\lambda+2r) r! (1+v)^{2\lambda+2r}} \\
 \text{or } &= \frac{e^{-\theta}}{\sqrt{\pi}} \sum_{r=0}^{\infty} \frac{2^{2\lambda-1} (2\theta)^{2r} \Gamma(\lambda+r+\frac{1}{2}) {}_1F_1(-\lambda; \lambda+2r; -\theta) v^{\lambda+r-1}}{\Gamma(\lambda+2r) r! (1+v)^{2\lambda+2r}} \dots (55)
 \end{aligned}$$

by the application of Kummer's transformation

$${}_1F_1(\alpha; \beta; x) = e^x {}_1F_1(\beta-\alpha; \beta; -x).$$

The corresponding result for (39) (when $\Delta=0$) is obtained by putting $\theta=0$ in (55) and is

$$P(v) = \frac{\Gamma(2\lambda)}{\{\Gamma(\lambda)\}^2} \cdot \frac{v^{\lambda-1}}{(1+v)^{2\lambda}} \dots (56)$$

If $\lambda = m+1$ is an integer, i.e., p , the number of variates, is an even integer, it may be noted that the hyper-geometric series in (55) will be a terminating one.

It is needless to mention that these results can easily be transformed into expressions containing the original parameters.

9. Allied Distributions for Pearson's Type III

Pearson's Type III curve may be regarded as a special case of the distribution considered here and is simply obtained by putting $a=1$ in the equation (39). This at once gives us

$$P(x) = \frac{1}{\Gamma(m+1)} e^{-x} x^m \dots (57)$$

Similarly, by applying the transformations (36) to (35), (44) and (56) respectively, we find that for this curve (57),

$$P(\bar{x}) = \frac{1}{\Gamma\{n(m+1)\}} e^{-n\bar{x}} \bar{x}^{mn+n-1} \quad \dots (58)$$

a result first given by Irwine,¹²

$$P(u) = \frac{u^{m+\frac{1}{2}} K_{m+\frac{1}{2}}(u)}{\sqrt{\pi} 2^{m+\frac{1}{2}} \Gamma(m+1)}; \quad \dots (59)$$

and
$$P(v) = \frac{\Gamma(2\lambda)}{\{\Gamma(\lambda)\}^2} \frac{v^{\lambda-1}}{(1+v)^{2\lambda}}, \quad \dots (60)$$

results which were first given by S. Kullback.¹³

Summary

On the assumption that the system of variances and covariances of the two multivariate normal populations concerned are known, the distribution of the D^2 -statistic has been obtained here by the method of characteristic functions. It has also been shown that the distribution of the mean of the k values of D^2 obtained from k independent random samples follows the same general law (with change of parameters). Both these distributions were first obtained by R. C. Bose, by other methods.

Further, the distribution laws of the difference and quotient of two independent values of D^2 from the same populations have been worked out.

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AN ALTERNATIVE METHOD OF THE DISTRIBUTION OF MAHALANOBIS'S D²-STATISTIC

By

B. C. BHATTACHARYYA

1. The exact distribution of Mahalanobis's D²-statistic has been first given by R. C. Bose (1936). He has also worked out the first four moments and β_1 , and β_2 , for the distribution, and shown them to agree with those originally given by Mahalanobis (1930) in the special case of populations with uncorrelated variates. Bose has followed geometrical method of derivation. The object of the present note is to derive the same results by employing "characteristic function"—a process which does not require any complicated geometrical notions to be introduced.

2. Let there be two multivariate populations π and π' , each with P variates, which are linearly correlated, and let Σ and Σ' be two random samples of sizes n and n' respectively, drawn from them. Following Bose's notations,

let α_p = mean of the p^{th} variate in population π

σ_p = S.D. " " " " "

ρ_{pq} = coeff. of correlation between the p^{th} and q^{th} variates in the population π

a_p = mean of the variates in sample Σ .

Let Λ_{pq} be the co-factor of a_{pq} in the determinant

$$\Lambda \equiv \begin{vmatrix} a_{11} & a_{12} \dots a_{1p} \\ a_{21} & a_{22} \dots a_{2p} \\ \dots \dots \dots \\ a_{p1} & a_{p2} \dots a_{pp} \end{vmatrix} \text{ where } a_{pq} = \sigma_p \sigma_q \rho_{pq}.$$

Let the same notations with dashes indicate the same quantities for the second population π' and the second sample drawn from it, Σ' .

Now let $\bar{a}_{pq} = \frac{n'a_{pq} + na'_{pq}}{n + n'}$ and let β_{pq} be the co-factor of $\bar{a}_{pq}$ in

the determinant

$$\beta \equiv \begin{vmatrix} \bar{a}_{11} & \bar{a}_{12} \dots \bar{a}_{1p} \\ \bar{a}_{21} & \bar{a}_{22} \dots \bar{a}_{2p} \\ \dots \dots \dots \\ \bar{a}_{p1} & \bar{a}_{p2} \dots \bar{a}_{pp} \end{vmatrix}$$

Then the distribution of the differences of the means from the two samples, namely $(a_1 - a'_1), (a_2 - a'_2), \dots, (a_p - a'_p)$, is given by $p[(a_1 - a'_1), (a_2 - a'_2), \dots, (a_p - a'_p)]$

$$= \text{const.} \exp \left\{ -\frac{\bar{n}}{4\beta} [\beta_{11}\{(a_1 - a'_1) - (a_1 - a'_1)\}^2 + \dots + 2\beta_{12}\{(a_1 - a'_1) - (a_2 - a'_2)\} + \dots] \right\} \dots (2.1)$$

Writing for brevity,

$$\text{and} \quad \left. \begin{aligned} x_p &= a_p - a'_p \\ \xi_p &= a_p - a'_p \end{aligned} \right\} \text{ where } p=1, 2, \dots, P,$$

equation (2.1) becomes

$$p(x_1, x_2, \dots, x_p) = \text{const.} \exp \left\{ -\frac{\bar{n}}{4\beta} [\beta_{11}(x_1 - \xi_1)^2 + \dots + 2\beta_{12}(x_1 - \xi_1)(x_2 - \xi_2) + \dots] \right\} \dots (2.2)$$

3. Mahalanobis's D^2 -statistic may now be defined as [see article 3 in Bose's Paper (1936)]

$$D^2 = D_1^2 - \frac{2}{n} \dots (3.1)$$

where

$$D_1^2 = \frac{1}{P\beta} [\beta_{11}x_1^2 + \beta_{22}x_2^2 + \dots + 2\beta_{12}x_1x_2 + 2\beta_{13}x_1x_3 + \dots] \dots (3.2)$$

We also set

$$\Delta^2 = \frac{1}{P\beta} [\beta_{11}\xi_1^2 + \dots + 2\beta_{12}\xi_1\xi_2 + \dots] \dots (3.3)$$

Normalising the variates by linear transformation as in Bose's Paper, we can have

$$\frac{1}{2\beta} [\beta_{11}x_1^2 + \dots + 2\beta_{12}x_1x_2 + \dots] = y_1^2 + y_2^2 + \dots + y_p^2 \dots (3.4)$$

And by a similar transformation of ξ 's, we can have

$$\frac{1}{2\beta} [\beta_{11}\xi_1^2 + \dots + 2\beta_{12}\xi_1\xi_2 + \dots] = \eta_1^2 + \eta_2^2 + \dots + \eta_p^2 \dots (3.5)$$

Then it follows that

$$\frac{1}{2\beta} [\beta_{11}x_1\xi_1 + \dots + \beta_{12}(x_1\xi_2 + x_2\xi_1) + \dots] = y_1\eta_1 + \dots + y_p\eta_p \dots (3.6)$$

The Distribution of y 's derived from (2.2) is therefore

$$p(y_1, y_2 \dots y_p) = \text{const.} \exp \left\{ -\frac{\bar{n}}{2} [(y_1 - \eta_1)^2 + \dots + (y_p - \eta_p)^2] \right\} \dots \quad (3.7)$$

The constant, obtained on integration, is easily seen to be $\left(\frac{\bar{n}}{2\pi}\right)^{\frac{1}{2}P}$.

$$\text{Also} \quad D_1^2 = \frac{2}{P} (y_1^2 + y_2^2 + \dots + y_p^2) \quad \dots \quad (3.8)$$

$$\text{and} \quad \Delta^2 = \frac{2}{P} (\eta_1^2 + \eta_2^2 + \dots + \eta_p^2). \quad \dots \quad (3.9)$$

4. It is observed from (3.7) that $y_1, y_2 \dots y_p$ are independent normal variates with means $\eta_1, \eta_2 \dots \eta_p$ and a common S.D. = $\sqrt{\frac{1}{\bar{n}}}$.

We shall now proceed to find the characteristic function of y_p^2 , of which the suffix will be dropped for convenience.

Characteristic function of a random variable x is defined to be

$$\phi_x(t) = \int \exp(itx) p(x) dx \quad \dots \quad (4.1)$$

where $p(x)$ is the elementary probability law of x , and the integration is to be performed over the whole range of x . Since

$$p(y) = \text{const.} \exp \left\{ -\frac{\bar{n}}{2} (y - \eta)^2 \right\}$$

$$\therefore p(y^2) = p(y) \frac{dy}{d(y^2)} = \text{const.} \cdot y^{-1} \exp \left\{ -\frac{\bar{n}}{2} (y - \eta)^2 \right\}.$$

$$\text{Hence} \quad \phi_{y^2}(t) = \text{const.} \int_0^\infty y^{-1} \exp \left\{ -\frac{\bar{n}}{2} (y - \eta)^2 \right\} \exp(it y^2) \cdot d(y^2),$$

(obviously y^2 changes from 0 to ∞)

which gives on integration,

$$\phi_{y^2}(t) = C. \exp \left\{ \frac{it\eta^2}{1 - \frac{2it}{\bar{n}}} \right\} / \left\{ 1 - \frac{2it}{\bar{n}} \right\}^{\frac{1}{2}}$$

Putting $t=0$,

$$\phi_{y^2}(t) = C.$$

Hence by division, remembering that $\phi_{y2}(0)=1$,

$$\phi_{y2}(t) = \exp \left\{ \frac{it\eta^2}{1 - \frac{2it}{n}} \right\} / \left\{ 1 - \frac{2it}{n} \right\}^{\frac{1}{2}}. \quad \dots (4.2)$$

The characteristic function of D_1^2 defined in (3.8) is therefore given by

$$\phi_{D_1^2}(t) = \phi_{2 \sum_{p=1}^p y_p^2}(t) = \prod_{p=1}^p \phi_{2y_p^2}(t),$$

since y 's are independent,

$$= \exp \left\{ \frac{it\Delta^2}{1 - \frac{4it}{Pn}} \right\} / \left\{ 1 - \frac{4it}{Pn} \right\}^{\frac{p}{2}}. \quad \dots (4.3)$$

5. The r^{th} moment of the distribution of D_1^2 about the origin is of course the coefficient of $\frac{(it)^r}{r!}$ in the expansion of the function (4.3).

But the moments about the mean of a distribution, particularly the first four, are more easily obtained from the cumulants which are given by the coefficients of the function

$$\psi_{D_1^2}(t) = \log \phi_{D_1^2}(t) = it\Delta^2 \left(1 - \frac{4it}{Pn} \right)^{-1} - \frac{P}{2} \log \left(1 - \frac{4it}{Pn} \right). \quad \dots (5.1)$$

The r^{th} cumulant, K_r , is the coefficient of $\frac{(it)^r}{r!}$ in the expansion of (5.1). Hence

$$K_r = \left[\left(\frac{4}{Pn} \right)^{r-1} \Delta^2 + \frac{P}{2r} \left(\frac{4}{Pn} \right)^r \right] \times r!. \quad \dots (5.2)$$

It follows that

$$\left. \begin{aligned} k_1 &= \Delta^2 + \frac{2}{n}, & k_2 &= \frac{8}{Pn} \left[\Delta^2 + \frac{1}{n} \right], \\ k_3 &= \frac{32}{P^2 n^2} \left[3\Delta^2 + \frac{2}{n} \right], & k_4 &= \frac{12.4^3}{P^2 n^3} \left[2\Delta^2 + \frac{1}{n} \right]. \end{aligned} \right\} \dots (5.3)$$

Remembering that

$k_1 = \mu_1 =$ the mean from the origin,

$k_2 = \mu_2 =$ the second moment about the mean,

$k_3 = \mu_3 =$ the third moment about the mean,

and

$k_4 = \mu_4^2 (\beta_2 - 3),$

and writing $\bar{n}\Delta^2 = \delta$ as in Bose's equations (9.21) to (9.6) we get the corresponding results that

$$\left. \begin{aligned} \mu_1 &= \frac{\delta+2}{n}, \quad \mu_2 = \frac{8}{Pn^2}(\delta+1), \\ \beta_1 &= \frac{2}{P}(3\delta+2)^2/(\delta+1)^3, \quad \beta_2 = 3 + \frac{12}{P}(2\delta+1)/(\delta+1)^2. \end{aligned} \right\} \dots (5.4)$$

Equations (5.4) give the coefficients with regard to the distribution of D_1^2 . The last three, however, are also true without any alteration in respect of the distribution of D^2 , defined in (3.1). The mean value of D^2 , i.e., μ_1 for the distribution of D^2 is

$$\begin{aligned} \mu_1 &= E(D^2) = E\left(D_1^2 - \frac{2}{n}\right) = E(D_1^2) - \frac{2}{n} \\ &= \Delta^2 + \frac{2}{n} - \frac{2}{n} = \Delta^2 \text{ [from (5.3)].} \end{aligned} \dots (5.5)$$

6. To obtain the distribution of D_1^2 , we make use of the Fourier integral theorem that if

$$\phi(x) = \int_{\text{whole range of } x} \exp(itx) p(x) dx,$$

then
$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-itx) \phi_x(t) dt.$$

The characteristic function of D_1^2 being given in (4.3) the distribution of D_1^2 is therefore given by

$$p(D_1^2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-itD_1^2) \frac{\exp\left\{\frac{it\Delta^2}{1 - \frac{4it}{Pn}}\right\}}{\left(1 - \frac{4it}{Pn}\right)^{\frac{P}{2}}} dt. \dots (6.1)$$

Applying the transformation $1 - \frac{4it}{Pn} = z$, the integral reduces to

$$\frac{Pn}{8\pi i} \exp\left\{-\frac{Pn}{4}(D_1^2 + \Delta^2)\right\} \int_{1-i\infty}^{1+i\infty} z^{-\frac{P}{2}} \exp\left\{\frac{Pn}{4}\left(\frac{\Delta^2}{z} + D_1^2 z\right)\right\} dz. \dots (6.2)$$

By the further transformation $z = \frac{\Delta}{\sqrt{x}} u$, it becomes

$$\frac{\bar{P}_n}{8\pi i} \exp \left\{ -\frac{\bar{P}_n}{4} (D_1^2 + \Delta^2) \right\} \left\{ \left(\frac{D_1^2}{\Delta^2} \right)^{\frac{P}{4} - \frac{1}{2}} \int_{c-i\infty}^{c+i\infty} u^{-\frac{P}{2}} \times \exp \left\{ \frac{\bar{P}_n}{4} \Delta D_1 \left(u + \frac{1}{u} \right) \right\} du \right\} \quad (6.3)$$

where $c = \frac{D_1}{\Delta}$. The above integral is easily seen to be equivalent to the contour integral

$$\frac{\bar{P}_n}{4} \exp \left\{ -\frac{\bar{P}_n}{4} (D_1^2 + \Delta^2) \right\} \left\{ \left(\frac{D_1^2}{\Delta^2} \right)^{\frac{P}{4} - \frac{1}{2}} \frac{1}{2\pi i} \int_{-\infty}^{(0+)} u^{-\frac{P}{2}} \times \exp \left\{ \frac{\bar{P}_n \Delta}{4} D_1 \left(u + \frac{1}{u} \right) \right\} du \right\} \quad (6.4)$$

Now it is known that

$$I_\nu(z) = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} u^{\nu-1} \exp \left\{ \frac{1}{2} z \left(u + \frac{1}{u} \right) \right\} du \quad (6.5)$$

Hence

$$p(D_1^2) = \frac{\bar{P}_n}{4} \exp \left\{ -\frac{\bar{P}_n}{4} (D_1^2 + \Delta^2) \right\} \left\{ \left(\frac{D_1^2}{\Delta^2} \right)^{\frac{P}{4} - \frac{1}{2}} I_{\frac{P}{2}} - \frac{1}{2} \bar{P}_n D_1 \Delta \right\}$$

which agrees with Bose's result (6.5) and finally his result (6.6) is secured.

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ON THE PROPERTIES OF THE FUNCTIONS WHICH ARE SELF-RECIPROCAL IN HANKEL'S TRANSFORM

By

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Bailey¹ has proved the property of two definite integrals containing self-reciprocal functions. Here I shall consider similar property of two more general definite integrals and certain new properties of self-reciprocal functions.

$$\text{Let } f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left[\frac{1}{4} + \frac{\mu'}{2} + \frac{s}{2} \right] 2^{\frac{s}{2}} \psi(s) ds$$

where $\psi(s) = \psi(1-s)$ in the strip $a < \sigma < 1-a$... (1)

$$\text{and } \psi(s) = 0 \left\{ e^{(\frac{\pi}{4} - \omega + \eta)t} \right\}$$

c being any value of σ in (1).

Then $f(x)$ is $R_{(\mu')}$ i.e., it is self-reciprocal in Hankel's Transform.

Let

$$\phi_{\mu}(y) = \int_0^{\infty} x^{\mu + \frac{1}{2}} e^{-\frac{y}{2}x^2 \cos \alpha} \cos\left(\frac{y}{2}x^2 \sin \alpha\right) f(x) dx$$

$$\psi_{\mu}(y) = \int_0^{\infty} x^{\mu + \frac{1}{2}} e^{-\frac{y}{2}x^2 \cos \alpha} \sin\left(\frac{y}{2}x^2 \sin \alpha\right) f(x) dx$$

$$\text{where } -\frac{1}{2} \leq \mu < \frac{1}{2}, \quad 0 < \alpha \leq \frac{\pi}{2}.$$

$$\text{when } \alpha = \frac{\pi}{2} \text{ we have Bailey's case.}^2$$

Now

$$\begin{aligned} \phi_{\mu}(y) - i\psi_{\mu}(y) &= \int_0^{\infty} x^{\mu + \frac{1}{2}} e^{-\frac{y}{2}x^2} e^{-i(\frac{\pi}{2} - \alpha)} f(x) dx \\ &= \int_0^{\infty} x^{\mu + \frac{1}{2}} e^{-\frac{y}{2}x^2} i e^{-i(\frac{\pi}{2} - \alpha)} dx \int_0^{\infty} f\left(\frac{y}{2}xt\right)^{\frac{1}{2}} J_{\mu}(xt) dt \end{aligned}$$

$$\begin{aligned}
&= \int_0^\infty f(t) t^{\frac{1}{2}} dt \int_0^\infty x^{\mu+1} e^{-\frac{1}{2}x^2} e^{-i(\frac{\pi}{2}-\alpha)} J_\mu(xt) dx \\
&= \int_0^\infty f(t) t^{\mu+\frac{1}{2}} x^{-\mu-1} e^{-\frac{1}{2}i(\mu+1)\frac{\pi}{2}} e^{-\frac{1}{2} \frac{t^2}{iy} e^{-i(\frac{\pi}{2}-\alpha)}} dt \\
&= \frac{e^{-\frac{1}{2}i(\mu+1)\pi}}{x^{\mu+1}} \int_0^\infty f(t) t^{\mu+\frac{1}{2}} dt \left[\cos\left(\frac{t^2 \sin \alpha}{2y}\right) \right. \\
&\quad \left. + i \sin\left(\frac{t^2 \sin \alpha}{2y}\right) \right] e^{-\frac{t^2 \cos \alpha}{2y}} \\
&= \frac{1}{x^{\mu+1}} \left[\cos(\mu+1)\frac{\pi}{2} - i \sin(\mu+1)\frac{\pi}{2} \right] \left[\phi_\mu\left(\frac{1}{y}\right) + i \psi_\mu\left(\frac{1}{y}\right) \right] \\
&\quad \text{Hence } \phi_\mu(y) = \frac{1}{x^{\mu+1}} \left[\cos(\mu+1)\frac{\pi}{2} \phi_\mu\left(\frac{1}{y}\right) + \sin(\mu+1)\frac{\pi}{2} \psi_\mu\left(\frac{1}{y}\right) \right] \\
&\quad \psi_\mu(y) = \frac{1}{x^{\mu+1}} \left[-\cos(\mu+1)\frac{\pi}{2} \psi_\mu\left(\frac{1}{y}\right) + \sin(\mu+1)\frac{\pi}{2} \phi_\mu\left(\frac{1}{y}\right) \right].
\end{aligned}$$

II

We shall now prove the justification of inversion. We have

$$\begin{aligned}
&\left| \int_0^\infty x^{\mu+\frac{1}{2}} e^{-\frac{1}{2}yx^2 \cos \alpha} \frac{\cos\left(\frac{y}{2}x^2 \sin \alpha\right)}{\sin\left(\frac{y}{2}x^2 \sin \alpha\right)} f(x) dx \right| \\
&< \int_0^\infty x^{\mu+\frac{1}{2}} |f(x)| dx, \text{ since } 0 \leq \alpha \leq \frac{\pi}{2}.
\end{aligned}$$

Now $\int_0^\infty x^{\mu+\frac{1}{2}} |f(x)| dx$ exists when $-\frac{1}{2} \leq \mu < \frac{1}{2}$.

$$\text{Hence } \left| \int_0^\infty x^{\mu+\frac{1}{2}} e^{-\frac{1}{2}yx^2 \cos \alpha} \frac{\cos\left(\frac{y}{2}x^2 \sin \alpha\right)}{\sin\left(\frac{y}{2}x^2 \sin \alpha\right)} f(x) dx \right|$$

exists and inversion is justifiable.

III

$$\text{Let } f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{\frac{s}{2}} \left[\frac{1}{4} + \frac{\mu}{2} + \frac{s}{2} \right] \psi(s) x^{-s} ds$$

then $f(x)$ is R_μ .

$$\begin{aligned}
 -x \frac{d}{dx} \{xf(x)\} &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{\frac{s}{2}} \left[\frac{1}{4} + \frac{\mu}{2} + \frac{s}{2} \right] s(1-s)\psi(s)x^{-s} ds \\
 &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{\frac{s}{2}} \left[\frac{1}{4} + \frac{\mu}{2} + \frac{s}{2} \right] \psi_1(s)x^{-s} ds
 \end{aligned}$$

where $\psi_1(s) = s(1-s)\psi(s) = \psi_1(1-s)$.

Hence $-x \frac{d}{dx} \{xf(x)\}$ will be $R_{(\mu)}$ when $f(x)$ is $R_{(\mu)}$ (i)

Similarly $- \left[x^2 \frac{d^2 g(x)}{dx^2} + xg(x) \right]$ is $R_{(\nu)}$ if $f(x)$ be $R_{(\mu)}$

where $^4 \quad g(x) = \frac{1}{x} \int_0^\infty P\left(\frac{y}{x}\right) f(y) dy. \quad \dots (ii)$

Let $f_1(x) = \left(\frac{1}{4} + \frac{\mu}{2} \right) f(x) - \frac{1}{2} x f'(x).$

Then $f_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{\frac{s}{2}} \left[\frac{1}{4} + \frac{\mu+2+s}{2} \right] \psi(s)x^{-s} ds$

and consequently $f_1(x)$ is $R_{(\mu+2)}$.

If $f_2(x) = \left[\frac{1}{4} + \frac{\mu+2}{2} \right] f_1(x) - \frac{1}{2} x f'_1(x)$

then $f_2(x)$ is $R_{(\mu+4)}$ if $f(x)$ be $R_{(\mu)}$.

Hence in general $f_m(x)$ is $R_{(\mu+2m)}$ if $f(x)$ be $R_{(\mu)}$.

If $^5 \quad f(x) = \frac{x^{\frac{1}{2}-\nu}}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\frac{1}{2}sx^2} s^{-(\nu+1)} \psi(s) ds$

then $f(x)$ will be $R_{(\nu)}$.

Let $\nu > 2m+1$ and $F(x) = x^{\nu-\frac{1}{2}} f(x).$

Then $2^m x^{\frac{1}{2}-(\nu-2m)} \frac{d^m F(x)}{dx^{2m}}$ will be $R_{(\nu-2m)}$ if $f(x)$ be $R_{(\nu)}$.

IV

We know ⁶

$$\frac{1}{2} \frac{\overline{(-m)} \overline{(-s)}}{\overline{(\rho+m+s)}} \\ = \sum_{r=0}^{\infty} \frac{(-1)^r \overline{(\rho-1+r)} \overline{\left[\frac{\rho+1}{2}+r\right]} \overline{(r-m)} \overline{(r-s)}}{\overline{(r+1)} \overline{\left[\left(\frac{\rho-1}{2}+r\right)\right]} \overline{(\rho+m+r)} \overline{(\rho+s+r)}}$$

Putting $m = -\frac{\mu+s+1+\frac{1}{4}}{2}$, $s = -\frac{\mu+s+\frac{1}{4}}{2}$, $\rho = \mu+s+\frac{3}{4}$

we have

$$\overline{\mu+s+\frac{1}{4}} = \frac{\overline{\left[\frac{1}{4}\right]}}{\sqrt{\pi}} 2^{\mu+s+\frac{1}{4}} \sum_{r=0}^{\infty} (-1)^r \overline{\left[\frac{\mu+s+\frac{1}{4}}{2}+r\right]} \overline{r}$$

Hence

$$\overline{\frac{\mu+s}{2}+\frac{1}{4}} = \frac{\overline{\left[\frac{1}{4}\right]}}{\sqrt{\pi}} 2^{\frac{\mu}{2}+\frac{1}{4}+\frac{s}{2}} \times \sum_{r=0}^{\infty} \frac{(-1)^r \overline{\left[\frac{\mu+s}{2}+\frac{1}{4}+r\right]} \overline{r}}{\overline{r}} \dots (A)$$

Therefore

$$\begin{aligned} f_{\mu}(x) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{\frac{s}{2}} \overline{\left[\frac{\mu+s}{2}+\frac{1}{4}\right]} x^{-s} \psi(s) ds \\ &= \frac{\overline{\left[\frac{1}{4}\right]}}{\sqrt{\pi}} \frac{2^{\frac{\mu}{2}+\frac{1}{4}}}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{\frac{s}{2}} x^{-s} \psi(s) ds \times \sum_{r=0}^{\infty} \frac{\overline{\left[\frac{\mu+s}{2}+\frac{1}{4}+r\right]} \overline{r}}{\overline{r}} (-1)^r \\ &= \frac{\overline{\left[\frac{1}{4}\right]}}{\sqrt{\pi}} 2^{\frac{\mu}{2}+\frac{1}{4}} \sum_{r=0}^{\infty} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{2s^{\frac{\mu+s}{2}+\frac{1}{4}+r}}{\overline{r}} \times x^{-s} \psi(s) ds (-1)^r \\ &= \frac{\overline{\left[\frac{1}{4}\right]}}{\sqrt{\pi}} 2^{\frac{\mu}{2}+\frac{1}{4}} \sum_{r=0}^{\infty} (-1)^r \frac{f_{\mu+2r}\left(\frac{x}{\sqrt{2}}\right)}{\overline{r}} \end{aligned}$$

where $f_m(x)$ is $R_{(m)}$.

It is clear that (A) is uniformly convergent within the range of integration and hence the order of summation and integration may be interchanged.

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PROPERTIES OF A CERTAIN CONFLUENT HYPER- GEOMETRIC FUNCTION

By

B. MOHAN

In a recent paper I have studied the properties of the function

$$\begin{aligned}\phi_\nu(x) &= \sum_0^\infty \frac{(-1)^\gamma \Gamma(\frac{1}{2} + \gamma) x^{\nu+2\gamma}}{2^{\nu+2\gamma} \Gamma(1+\nu+\gamma) \gamma} \\ &= \frac{\Gamma(\frac{1}{2})}{\Gamma(1+\nu)} \left(\frac{x}{2}\right)^\nu {}_1F_1\left(\frac{1}{2}; 1+\nu; -\left(\frac{x}{2}\right)^2\right) \quad \text{Re}(\nu) > -1.\end{aligned}$$

This has led me to study, in this note, the properties of the analogous function

$$\psi_\nu(x) = \sum_0^\infty \frac{\Gamma(\frac{1}{2} + \gamma) x^{\nu+2\gamma}}{2^{\nu+2\gamma} \Gamma(1+\nu+\gamma) \gamma} \quad \dots \quad (1.1)$$

To begin with, the series (1.1) is absolutely and uniformly convergent in any finite interval of values of x . It is also obvious that

$$\psi_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{x}} e^{\frac{1}{4}x^2}, \quad \dots \quad (1.2)$$

$$\psi_\nu(x) = \frac{(\frac{1}{2}x)^\nu \sqrt{\pi}}{\Gamma(\nu+1)} {}_1F_1\left(\frac{1}{2}; \nu+1; \frac{1}{4}x^2\right). \quad \dots \quad (1.3)$$

2. Differentiating (1.1) with respect to x , we have

$$\begin{aligned}\psi'_\nu(x) &= \sum_0^\infty \frac{\Gamma(\frac{1}{2} + \gamma)(\nu+2\gamma)x^{\nu+2\gamma-1}}{2^{\nu+2\gamma} \gamma \Gamma(\nu+\gamma+1)} \\ &= \sum_0^\infty \frac{\Gamma(\frac{1}{2} + \gamma)(2\nu+2\gamma)x^{\nu+2\gamma-1}}{2^{\nu+2\gamma} \gamma \Gamma(\nu+\gamma+1)} - \nu \sum_0^\infty \frac{\Gamma(\frac{1}{2} + \gamma)x^{\nu+2\gamma-1}}{2^{\nu+2\gamma} \gamma \Gamma(\nu+\gamma+1)} \\ &= \sum_0^\infty \frac{\Gamma(\frac{1}{2} + \gamma)x^{\nu+2\gamma-1}}{2^{\nu+2\gamma-1} \gamma \Gamma(\nu+\gamma)} - \frac{\nu}{x} \psi_\nu(x).\end{aligned}$$

Thus, we find that

$$\psi'_\nu(x) = \psi_{\nu-1}(x) - \frac{\nu}{x} \psi_\nu(x). \quad \dots \quad (2.1)$$

Again,

$$\begin{aligned}
 & \left(\frac{x}{2} + \frac{v}{x} \right) \psi_v(x) \\
 &= \left(\frac{x}{2} + \frac{v}{x} \right) \sum_0^{\infty} \frac{\Gamma(\frac{1}{2} + \gamma) x^{v+2\gamma}}{2^{v+2\gamma} \underline{\Gamma}(\nu + \gamma + 1)} \\
 &= \sum_0^{\infty} \frac{\Gamma(\frac{1}{2} + \gamma) x^{v+2\gamma+1}}{2^{v+2\gamma+1} \underline{\Gamma}(\nu + \gamma + 1)} + v \sum_0^{\infty} \frac{\Gamma(\frac{1}{2} + \gamma) x^{v+2\gamma-1}}{2^{v+2\gamma} \underline{\Gamma}(\nu + \gamma + 1)} \\
 &= \sum_0^{\infty} \frac{\Gamma(\frac{1}{2} + \gamma) x^{v+2\gamma+1}}{2^{v+2\gamma+1} \underline{\Gamma}(\nu + \gamma + 1)} + \sum_0^{\infty} \frac{\Gamma(\frac{1}{2} + \gamma)(v+2\gamma) x^{v+2\gamma-1}}{2^{v+2\gamma} \underline{\Gamma}(\nu + \gamma + 1)} \\
 &\quad - \sum_1^{\infty} \frac{\Gamma(\frac{1}{2} + \gamma) x^{v+2\gamma-1}}{2^{v+2\gamma-1} \underline{\Gamma}(\nu + \gamma + 1)} \\
 &= \psi_v'(x) + \sum_0^{\infty} \frac{\Gamma(\frac{1}{2} + \gamma) x^{v+2\gamma+1}}{2^{v+2\gamma+1} \underline{\Gamma}(\nu + \gamma + 1)} - \sum_0^{\infty} \frac{\Gamma(\frac{3}{2} + \gamma) x^{v+2\gamma+1}}{2^{v+2\gamma+1} \underline{\Gamma}(\nu + \gamma + 2)} \\
 &= \psi_v'(x) + \sum_0^{\infty} \frac{\Gamma(\frac{1}{2} + \gamma) x^{v+2\gamma+1}}{2^{v+2\gamma+1} \underline{\Gamma}(\nu + \gamma + 1)} \left(1 - \frac{\frac{1}{2} + \gamma}{\nu + \gamma + 1} \right).
 \end{aligned}$$

Hence we get

$$\left(\frac{x}{2} + \frac{v}{x} \right) \psi_v(x) = \psi_v'(x) + \left(v + \frac{1}{2} \right) \psi_{v+1}(x). \quad \dots (2.2)$$

As a particular case,

$$2\psi_0'(x) = x\psi_0(x) - \psi_1(x). \quad \dots (2.3)$$

Now, from (2.1) we have

$$\psi_v'' = \psi_{v-1}' - \frac{v}{x} \psi_v' + \frac{v}{x^2} \psi_v.$$

Also, from (2.2),

$$\left(\frac{x}{2} + \frac{v-1}{x} \right) \psi_{v-1} = \psi_{v-1}' + \left(v - \frac{1}{2} \right) \psi_v.$$

$\therefore$ subtracting, we have

$$\psi_v'' - \left(\frac{x}{2} + \frac{v-1}{x} \right) \psi_{v-1} = -\frac{v}{x} \psi_v' + \left(\frac{v}{x^2} - v + \frac{1}{2} \right) \psi_v.$$

That is,

$$\psi_v'' - \left(\frac{x}{2} + \frac{v-1}{x} \right) \left(\psi_v' + \frac{v}{x} \psi_v \right) + \frac{v}{x} \psi_v' + \left(v - \frac{1}{2} - \frac{v}{x^2} \right) \psi_v = 0,$$

on using (2.1). Thus we get

$$\psi_v'' + \left(\frac{1}{x} - \frac{x}{2} \right) \psi_v' + \left(\frac{1}{2}v - \frac{1}{2} - \frac{v^2}{x^2} \right) \psi_v = 0. \quad \dots (2.4)$$

This result could also be arrived at by starting with (1.3) and making use of the differential equation satisfied by the function ${}_1F_1$.

Other results that can be easily derived from the above formulae are :

$$\left(1 + \frac{x^2}{2v} \right) \psi_v - \left(2 + \frac{x^2}{2v} \right) \psi_v' = \left(v + \frac{1}{2} \right) \psi_{v+1} \quad \dots (2.5)$$

$$\left(\frac{x}{2} + \frac{2v}{x} \right) \psi_v - \psi_{v-1} = \left(v + \frac{1}{2} \right) \psi_{v+1} \quad \dots (2.6)$$

$$\psi_v'' = \psi_{v-2} - \frac{2v-1}{x} \psi_{v-1} + \frac{v(v+1)}{x^2} \psi_v \quad \dots (2.7)$$

$$\left(\frac{d}{x dx} \right)^m \left(x^v \psi_v \right) = x^{v-m} \psi_{v-m}, \quad \dots (2.8)$$

where m is a positive integer.

$$\begin{aligned} \frac{d}{x dx} \left(x^{-v} \psi_v \right) &= \frac{1}{2} x^{-v} \psi_v - \left(v + \frac{1}{2} \right) x^{-v-1} \psi_{v+1} \\ \left(\frac{d}{x dx} \right)^2 \left(x^{-v} \psi_v \right) &= \frac{1}{4} x^{-v} \psi_v - \left(v + \frac{1}{2} \right) x^{-v-1} \psi_{v+1} \\ &\quad + \left(v + \frac{1}{2} \right) \left(v + \frac{3}{2} \right) x^{-v-2}. \end{aligned}$$

$$\begin{aligned} 3. \quad \psi_v(x) &= \sum_0^{\infty} \frac{\Gamma(\frac{1}{2} + \gamma) x^{v+2\gamma}}{2^{v+2\gamma} \Gamma(\frac{1}{2} + \gamma) \Gamma(1 + v + \gamma)} \\ &= \sum_0^{\infty} \frac{(\frac{1}{2}x)^v}{\Gamma(\frac{1}{2} + v) \Gamma(\frac{1}{2} + \gamma)} \cdot \frac{x^{2\gamma}}{2^{2\gamma}} \cdot \frac{\Gamma(\frac{1}{2} + v) \Gamma(\frac{1}{2} + \gamma)}{\Gamma(1 + v + \gamma)} \\ &= \sum_0^{\infty} \frac{(\frac{1}{2}x)^v}{\Gamma(\frac{1}{2} + v) \Gamma(\frac{1}{2} + \gamma)} \cdot \frac{x^{2\gamma}}{2^{2\gamma}} \int_0^1 t^{v-\frac{1}{2}} (1-t)^{\gamma-\frac{1}{2}} dt \\ &= \frac{(\frac{1}{2}x)^v}{\Gamma(\frac{1}{2} + v)} \int_0^1 t^{v-\frac{1}{2}} \sum_0^{\infty} \frac{x^{2\gamma}}{2^{2\gamma} \Gamma(\frac{1}{2} + \gamma)} (1-t)^{\gamma-\frac{1}{2}} dt, \end{aligned}$$

a process easy to justify.

Hence

$$\begin{aligned}\psi_\nu(x) &= \frac{(\frac{1}{2}x)^\nu}{\Gamma(\nu+\frac{1}{2})} \int_0^1 \frac{t^{\nu-\frac{1}{2}}}{\sqrt{1-t}} e^{\frac{1}{4}x^2(1-t)} dt & [R(\nu) > -\frac{1}{2}] \\ &= \frac{(\frac{1}{2}x)^\nu}{\Gamma(\nu+\frac{1}{2})} \int_0^1 \frac{(1-u)^{\nu-\frac{1}{2}}}{\sqrt{u}} e^{\frac{1}{4}x^2u} du. & \dots (3.1)\end{aligned}$$

These results may also be thrown into the forms

$$\begin{aligned}\psi_\nu(2\sqrt{\omega}) &= \frac{\omega^{\frac{1}{2}\nu}}{\Gamma(\nu+\frac{1}{2})} \int_0^1 \frac{t^{\nu-\frac{1}{2}}}{\sqrt{1-t}} e^{\omega(1-t)} dt \\ &= \frac{\omega^{\frac{1}{2}\nu}}{\Gamma(\nu+\frac{1}{2})} \int_0^1 \frac{(1-u)^{\nu-\frac{1}{2}}}{\sqrt{u}} e^{\omega u} du.\end{aligned}$$

(3.1) is also equivalent to

$$\psi_\nu(x) = 2 \frac{(\frac{1}{2}x)^\nu}{\Gamma(\nu+\frac{1}{2})} \int_0^{\frac{\pi}{2}} \sin^{2\nu}\theta e^{\frac{1}{4}x^2 \cos^2\theta} d\theta.$$

Finally, from (3.1), we have

$$\begin{aligned}\psi_\nu(x) &= \frac{x^{-\nu-1} e^{\frac{1}{4}x^2}}{2^\nu \Gamma(\nu+\frac{1}{2})} \int_0^{x^2} u^{\nu-\frac{1}{2}} \left(1-\frac{u}{x^2}\right)^{-\frac{1}{2}} e^{-\frac{1}{4}u} du \\ &\sim x^{-\nu-1} e^{\frac{1}{4}x^2} \quad \text{for large } x.\end{aligned}$$

4. Let us evaluate the integral

$$I = \int_0^\infty x^{p-1} e^{-ax^2} \psi_\nu(bx) dx,$$

where $4R(a) > R(b^2) > 0$; $R(\nu+p) > 0$. We have

$$\begin{aligned}I &= \int_0^\infty x^{p-1} e^{-ax^2} \sum_0^\infty \frac{\Gamma(\frac{1}{2}+\gamma)(bx)^{\nu+2\gamma}}{2^{\nu+2\gamma} \Gamma(\nu+\gamma+1)} dx \\ &= \sum_0^\infty \frac{\Gamma(\frac{1}{2}+\gamma)b^{\nu+2\gamma}}{2^{\nu+2\gamma} \Gamma(\nu+\gamma+1)} \int_0^\infty e^{-ax^2} x^{p+\nu+2\gamma-1} dx,\end{aligned}$$

a process easy to justify. Hence

$$\begin{aligned}I &= \sum_0^\infty \frac{\Gamma(\frac{1}{2}+\gamma)b^{\nu+2\gamma}}{2^{\nu+2\gamma+1} \Gamma(\nu+\gamma+1)} \frac{\Gamma(\frac{1}{2}p+\frac{1}{2}\nu+\gamma)}{\Gamma(1+\nu+\gamma)a^{\frac{1}{2}p+\frac{1}{2}\nu+\gamma}} \\ &= \frac{b^\nu}{2^{\nu+1} a^{\frac{1}{2}p+\frac{1}{2}\nu}} \sum_0^\infty \frac{(\frac{1}{2}p+\frac{1}{2}\nu, \gamma) \Gamma(\frac{1}{2}p+\frac{1}{2}\nu) \sqrt{\pi} (\frac{1}{2}, \gamma)}{(\nu+1, \gamma) \Gamma(1+\nu)} \left(\frac{b^2}{4a}\right)^\gamma.\end{aligned}$$

Thus, we find that

$$\int_0^\infty x^{p-1} e^{-ax^2} \psi_\nu(bx) dx = \frac{b^\nu \Gamma(\frac{1}{2}p + \frac{1}{2}\nu) \sqrt{\pi}}{2^{\nu+1} a^{\frac{1}{2}p + \frac{1}{2}\nu} \Gamma(\nu+1)} F\left(\frac{1}{2}, \frac{1}{2}p + \frac{1}{2}\nu; \nu+1; \frac{b^2}{4a}\right), \dots \quad (4.1)$$

where $4R(a) > R(b^2) > 0$; $R(\nu+p) > 0$.

Particular Cases

(i) $p = \nu + 2$

$$\int_0^\infty x^{\nu+1} e^{-ax^2} \psi_\nu(bx) dx = \frac{b^\nu \sqrt{\pi}}{2^\nu a^{\nu+\frac{1}{2}} \sqrt{4a-b^2}}, \dots \quad (4.2)$$

where $4R(a) > R(b^2) > 0$; $R(\nu) > -1$.

Further, putting $a = \frac{1}{2}b^2$, we get

$$\int_0^\infty x^{\nu+1} e^{-\frac{1}{2}b^2x^2} \psi_\nu(bx) dx = \frac{\sqrt{2\pi}}{b^{\nu+\frac{1}{2}}}, \dots \quad (4.3)$$

where $R(b) > 0$; $R(\nu) > -1$.

In particular, when $b=1$, we have

$$\int_0^\infty x^{\nu+1} e^{-\frac{1}{2}x^2} \psi_\nu(bx) dx = \sqrt{2\pi} \dots \quad (4.4)$$

(ii) $p=1$.

$$\int_0^\infty e^{-ax^2} \psi_\nu(bx) dx = \frac{b^\nu \Gamma(\frac{1}{2} + \nu) \sqrt{\pi}}{2^{\nu+1} a^{\frac{1}{2} + \nu} \Gamma(\nu+1)} F\left(\frac{1}{2}, \frac{1}{2} + \nu; 1 + \nu; \frac{b^2}{4a}\right) \dots \quad (4.5)$$

where $R(a) > R(b^2) > 0$; $R(\nu) > -1$.

If, in (4.1), we put $\nu = -\frac{1}{2}$ and make use of (1.2) we arrive at the familiar result

$$\int_0^\infty x^{p-\frac{3}{2}} e^{-(a-\frac{1}{4}b^2)x^2} dx = \frac{2^{p-\frac{3}{2}} \Gamma(\frac{1}{2}p - \frac{1}{4})}{(4a-b^2)^{\frac{1}{2}p - \frac{1}{4}}},$$

where $R(p) > \frac{1}{2}$; $4R(a) > R(b^2) > 0$.

DYADICS AND MULTIDYADICS IN HYPERSPACE

By

N. N. GHOSH

The theory of linear vector functions in three-dimensional space was developed in terms of 'dyadics' by J. W. Gibbs.¹ In spaces of higher dimensions greater part of his theory remains true and the development is in general *parallel* to the matrix theory. In a recent paper² I. M. Hostetter has generalized some theorems on dyadics by an extension of Gibbs' notations. Earlier, in a paper³ entitled 'Four vector algebra and analysis' C. E. Weatherburn considered dyadics formed with vectors of four components. The object of the present paper is to generalize the theory of dyadics in such a way that it *includes* the matrix theory and that it possesses greater comprehensiveness.

To suit our purpose some changes in the existing notations are introduced. The notation⁴ ab for a dyadic is modified by the introduction of dummy suffixes obeying the summation convention of the tensor calculus. The scalar product $a \cdot \rho$ of two vectors a and ρ is denoted by the symbol $\left\{ \begin{smallmatrix} a \\ \rho \end{smallmatrix} \right\}$ in conformity with the symbol for some higher order entity called scalar determinant⁵ of two sets of vectors. A transformation of the vector ρ by the dyadic ab is thus represented in the form

$$\rho' = a_i \left\{ \begin{smallmatrix} b_i \\ \rho \end{smallmatrix} \right\} \text{ or } \left\{ \begin{smallmatrix} b_i \\ \rho \end{smallmatrix} \right\} a_i, \quad \dots (1)$$

where i runs over the values $1, 2, \dots, n$.

In §§ 2-6 most of the known theorems on dyadics have been generalized from the present point of view. In § 7 the notion of multidyadics defining a linear transformation of multivectors is introduced and some general relations obtained. In this connection the treatment includes the theory of compound matrices as a particular case.

2. Complete and incomplete dyadics

Let the set of vectors a_i in (1) be referred to a system of n rectangular axes. If the scalar determinant

$$\Delta_a = \begin{vmatrix} a_1 & a_2 & \dots & a_n \\ a_1 & a_2 & \dots & a_n \end{vmatrix} \quad \dots \quad (2)$$

does not vanish, the system a_i will form a basis for the n -space.

An allied system of n vectors $a_{(h)}$ may now be defined by means of the equations

$$a_{(h)} = a_i \left\{ \begin{matrix} a_i \\ a_h \end{matrix} \right\} \mu / \Delta_a, \quad (h=1, 2, \dots, n), \quad \dots \quad (3)$$

where the suffix μ attached to the minor of Δ_a indicates that its complementary minor is to be taken. We have then

$$\left\{ \begin{matrix} a_i \\ a_{(h)} \end{matrix} \right\} = \begin{cases} 0, & i \neq h \\ 1, & i = h. \end{cases} \quad \dots \quad (4)$$

The system of vectors $a_{(h)}$ will be called the *co-vector system* allied to a_i .

The *identical dyadic* is now representable in either of the forms

$$a_i a_{(i)} \text{ or } a_{(i)} a_i. \quad \dots \quad (5)$$

Let b_i in (1) be a set of n linearly independent vectors, then the scalar determinant

$$\Delta_b^a = \begin{vmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{vmatrix} \quad \dots \quad (6)$$

will not vanish. The co-vector system allied to b_i may then be defined by

$$b_{(h)} = a_i \left\{ \begin{matrix} a_i \\ b_h \end{matrix} \right\} \mu / \Delta_b^a. \quad \dots \quad (7)$$

Any complete dyadic $c_h d_h$, for instance, is expressible in either of the forms

$$a_i \left\{ \begin{matrix} a_{(i)} \\ c_h \end{matrix} \right\} d_h \text{ or } a_{(i)} \left\{ \begin{matrix} a_i \\ c_h \end{matrix} \right\} d_h,$$

so that for a complete dyadic we have two typical forms

$$(i) \quad a_i b_{(i)}, \quad (ii) \quad a_{(i)} b_i. \quad \dots \quad (8)$$

In the standard forms (8), the *antecedents* being the same for all dyadics, the *consequents* alone determine the character of dyadics.

If the set $b_{(i)}$ or b_i in (8) consists of vectors not all linearly independent, the dyadic is called 'incomplete.' Let us consider the form 8(i) and suppose that only r of the vectors $b_{(i)}$ are mutually independent. Denoting them by $b_{(k)}$ ($k=1, 2, \dots, r$) let us form the scalar determinant

$$\bar{\Delta}_b = \begin{vmatrix} b_{(1)} b_{(2)} \dots b_{(r)} \\ b_{(1)} b_{(2)} \dots b_{(r)} \end{vmatrix} \quad \dots \quad (9)$$

A co-vector system allied to $b_{(k)}$ may then be defined by means of the equations

$$\bar{b}_k = b_{(j)} \begin{vmatrix} b_{(j)} \\ b_{(k)} \end{vmatrix}_\mu / \bar{\Delta}_b, \quad \dots \quad (10)$$

where j runs over the values $1, 2, \dots, r$.

The incomplete dyadic $a_i b_{(i)}$ is then expressible in the form

$$a_k b_{(k)} + a_p \begin{vmatrix} b_{(p)} \\ \bar{b}_k \end{vmatrix} b_{(k)}, \quad \dots \quad (11)$$

where p runs over the values $r+1, r+2, \dots, n$.

Denoting

$$\bar{a}_k + a_p \begin{vmatrix} b_{(p)} \\ \bar{b}_k \end{vmatrix}$$

by $\bar{a}_k$, the incomplete dyadic of type 8(i), has the standard form

$$\bar{a}_k b_{(k)}. \quad \dots \quad (12)$$

3. Product of dyadics

Let $a_i b_{(i)}$ and $a_h c_{(h)}$ denote two complete dyadics in the standard form 8(i). Then the dyadic

$$a_i \begin{vmatrix} b_{(i)} \\ a_h \end{vmatrix} c_{(h)}, \quad \dots \quad (13)$$

whose consequent is $c_{(h)} \begin{vmatrix} a_h \\ b_{(i)} \end{vmatrix}$ represents the product when $a_i b_{(i)}$ is

the prefactor. We shall use the symbol $(cb)_{(i)}$ to denote the consequent of (13).

The product of three dyadics $a, b_{(i)}$, $a, c_{(i)}$ and $a, d_{(i)}$ in successive order is represented in the form

$$a, \left\{ \begin{matrix} b_{(i)} \\ a_{i_1} \end{matrix} \right\} \left\{ \begin{matrix} c_{(i)} \\ a_{i_2} \end{matrix} \right\} d_{(i_2)} \text{ or } a, (dcb)_{(i)}. \quad \dots (14)$$

The *resolved form* of a dyadic is obtained by forming product with identical dyadic.

The resolved form of $a, b_{(i)}$, when complete, is given by

$$a, \left\{ \begin{matrix} b_{(i)} \\ a_h \end{matrix} \right\} a_{(h)} \quad \dots (15)$$

and, when incomplete, by

$$a, \left\{ \begin{matrix} a_{(i)} \\ a_k \end{matrix} \right\} \left\{ \begin{matrix} b_{(k)} \\ a_h \end{matrix} \right\} a_{(h)} \quad \dots (16)$$

as follows from (12)

The formulae (15) and (16) are referred to the basis a_i . For a change of basis, for instance, into A_i , the formula (15) will be of the form

$$A_{i_1} \left\{ \begin{matrix} A_{(i_1)} \\ a_1 \end{matrix} \right\} \left\{ \begin{matrix} b_{(i)} \\ a_h \end{matrix} \right\} \left\{ \begin{matrix} a_{(h)} \\ A_{i_2} \end{matrix} \right\} A_{(i_2)}. \quad \dots (17)$$

The theory of matrices is associated with such resolved forms of dyadics. We can represent (15) as a matrix of order n , whose element in the i th row and h th column is $\left\{ \begin{matrix} b_{(i)} \\ a_h \end{matrix} \right\}$. For a change of basis the matrix representation is given by (17). Properties of dyadics may thus be translated into the properties of matrices. It may be noted that incomplete dyadics introduce either degenerate or rectangular matrices.

The product of s dyadics each equal to $a, b_{(i)}$ will be represented by $a, b_{(i)}^s$. We have then

$$a, \left\{ \begin{matrix} b_{(i)}^s \\ a_h \end{matrix} \right\} b_{(h)}^t = a, b_{(i)}^{s+t} \text{ for any positive integers } s, t. \quad \dots (18)$$

The product of $a, b_{(i)}$ and $b, a_{(i)}$ being the identical dyadic we call $b, a_{(i)}$, the dyadic *reciprocal* to $a, b_{(i)}$. Expressed in the standard form 8(i), its consequent is given by

$$a_{(h)} \left\{ \begin{matrix} b_h \\ a_{(i)} \end{matrix} \right\}, \quad \dots (19)$$

which defines $b_{(i)}^{-1}$. We also set $b_{(i)}^0 = a_{(i)}$.

The reciprocal of $a, b_{(i)}^s$ is $b, a_{(i)}$, which is equivalent to $a, b_{(i)}^{-s}$.

4. Conjugate dyadics

The dyadic $b_{(i)}, a_i$ is said to be conjugate to $a, b_{(i)}$. Expressed in the standard form 8(ii), the consequent of the conjugate dyadic becomes

$$a_h \left\{ \begin{matrix} b_{(h)} \\ a_i \end{matrix} \right\},$$

which is the co-vector system allied to $b_{(i)}^{-1}$ in (19). We therefore represent it by b_i^{-1} . The conjugate of $a, b_{(i)}^s$ will be represented by $a_{(i)} b_i^{-s}$, where

$$b_i^{-s} = a_h \left\{ \begin{matrix} b_{(h)}^s \\ a_i \end{matrix} \right\}. \quad \dots (20)$$

The conjugate of the product of two dyadics in (13) is given by

$$(cb)_{(i)}, a_i \text{ or } a_{(i)}, (b^{-1}c^{-1})_i. \quad \dots (21)$$

Similarly the conjugate of the product of three dyadics in (14) is expressible in the form

$$a_{(i)}(b^{-1}c^{-1}d^{-1})_i. \quad \dots (22)$$

The dyadic $a, b_{(i)}^s + a_{(i)} b_i^{-s}$ is self-conjugate. Expressed in the standard form 8(i), its consequent becomes

$$b_{(i)}^s + \left\{ \begin{matrix} a_{(i)} \\ a_{(h)} \end{matrix} \right\} b_h^{-s}. \quad \dots (23)$$

The dyadic whose consequent is

$$b_{(i)}^s - \left\{ \begin{matrix} a_{(i)} \\ a_{(h)} \end{matrix} \right\} b_h^{-s} \quad \dots (24)$$

is anti-self conjugate.

5. Characteristic equation of a dyadic

Let us find a vector ρ such that

$$a_i \left\{ \begin{matrix} b_{(i)} \\ \rho \end{matrix} \right\} = l_i \rho \quad (l_i = \text{scalar}). \quad \dots \quad (25)$$

The above is equivalent to n linear equations

$$\left\{ \begin{matrix} b_{(i)} \\ a_h \end{matrix} \right\} \left\{ \begin{matrix} a_{(h)} \\ \rho \end{matrix} \right\} = l_i \left\{ \begin{matrix} a_{(i)} \\ \rho \end{matrix} \right\},$$

Eliminating $\left\{ \begin{matrix} a_{(i)} \\ \rho \end{matrix} \right\}$'s we get the equation

$$\left\{ \begin{matrix} b_{(1)} - l a_{(1)} & b_{(2)} - l a_{(2)} & \dots & b_{(n)} - l a_{(n)} \\ a_1 & a_2 & & a_n \end{matrix} \right\} = 0 \quad \dots \quad (26)$$

giving n values for l .

The above is called the characteristic equation of the dyadic $a_i b_{(i)}$. Applying the properties of scalar determinants the equation (26) may be expressed in the form

$$\begin{aligned} & \left\{ \begin{matrix} b_{(1)} \dots b_{(n)} \\ a_1 \dots a_n \end{matrix} \right\} - l \left[\left\{ \begin{matrix} b_{(1)} \\ a_1 \end{matrix} \right\}_\mu + \left\{ \begin{matrix} b_{(2)} \\ a_2 \end{matrix} \right\}_\mu + \dots + \left\{ \begin{matrix} b_{(n)} \\ a_n \end{matrix} \right\}_\mu \right] \\ & + l^2 \left[\left\{ \begin{matrix} b_{(1)} & b_{(2)} \\ a_1 & a_2 \end{matrix} \right\}_\mu + \left\{ \begin{matrix} b_{(1)} & b_{(3)} \\ a_1 & a_3 \end{matrix} \right\}_\mu + \dots + \left\{ \begin{matrix} b_{(n-1)} & b_{(n)} \\ a_{n-1} & a_n \end{matrix} \right\}_\mu \right] + \dots \\ & + (-1)^{n-1} l^{n-1} \left[\left\{ \begin{matrix} b_{(1)} \\ a_1 \end{matrix} \right\} + \left\{ \begin{matrix} b_{(2)} \\ a_2 \end{matrix} \right\} + \dots + \left\{ \begin{matrix} b_{(n)} \\ a_n \end{matrix} \right\} \right] + (-1)^n l^n = 0, \quad \dots \quad (27) \end{aligned}$$

where the suffix μ has the same significance as in (3) with regard to the scalar determinant

$$\left\{ \begin{matrix} b_{(1)} & b_{(2)} \dots b_{(n)} \\ a_1 & a_2 \dots a_n \end{matrix} \right\}.$$

Let the roots of (27) be all real and distinct. Corresponding to a root l_h let the vector ρ_h satisfy (25). We may then choose the set of n vectors $\rho_1, \rho_2, \dots, \rho_n$ as forming a new basis. Denoting the co-

vector system allied to ρ_h by $\rho_{(h)}$ an arbitrary vector ξ admits of being expressed in the form

$$\xi = \rho_h \left\{ \begin{matrix} \rho_{(h)} \\ \xi \end{matrix} \right\},$$

whence

$$a_i \left\{ \begin{matrix} b_{(i)} \\ \xi \end{matrix} \right\} = a_i \left\{ \begin{matrix} b_{(i)} \\ \rho_h \end{matrix} \right\} \left\{ \begin{matrix} \rho_{(h)} \\ \xi \end{matrix} \right\} = \sum_{h=1}^n l_h \rho_h \left\{ \begin{matrix} \rho_{(h)} \\ \xi \end{matrix} \right\}.$$

Thus the dyadic $a_i b_{(i)}$ reduces to the canonical form

$$l_1 \rho_1 \rho_{(1)} + l_2 \rho_2 \rho_{(2)} + \dots + l_n \rho_n \rho_{(n)}. \quad \dots (28)$$

For a pair of imaginary roots $l_x + il_y$, $l_x - il_y$ of (27) it is possible to find a pair of vectors ρ_x and ρ_y such that

$$a_i \left\{ \begin{matrix} b_{(i)} \\ \rho_x \end{matrix} \right\} = l_x \rho_x - l_y \rho_y,$$

$$a_i \left\{ \begin{matrix} b_{(i)} \\ \rho_y \end{matrix} \right\} = l_y \rho_x + l_x \rho_y.$$

These vectors ρ_x , ρ_y together with the rest will form a basis. Hence for every pair of imaginary roots of (27), the corresponding terms in (28) are replaced by

$$(l_x \rho_x - l_y \rho_y) \rho_{(x)} + (l_y \rho_x + l_x \rho_y) \rho_{(y)}$$

$$\text{or} \quad l_x (\rho_x \rho_{(x)} + \rho_y \rho_{(y)}) + l_y (\rho_x \rho_{(y)} - \rho_y \rho_{(x)}). \quad \dots (29)$$

Again, for a multiple root l_x of order r , a set of r vectors ρ_{x_1} , ρ_{x_2} , $\dots$, ρ_{x_r} may be so determined that

$$a_i \left\{ \begin{matrix} b_{(i)} \\ \rho_{x_1} \end{matrix} \right\} = l_x \rho_{x_1},$$

$$a_i \left\{ \begin{matrix} b_{(i)} \\ \rho_{x_2} \end{matrix} \right\} = l_x \rho_{x_2} + \rho_{x_1},$$

.....

$$a_i \left\{ \begin{matrix} b_{(i)} \\ \rho_{x_r} \end{matrix} \right\} = l_x \rho_{x_r} + \rho_{x_{r-1}}.$$

These vectors together with the rest will form a basis. Hence corresponding terms in (28) will be replaced by

$$l_x \left(\rho_{x_1} \rho_{(x_1)} + \rho_{x_2} \rho_{(x_2)} + \dots + \rho_{x_r} \rho_{(x_r)} \right) + \rho_{x_1} \rho_{(x_2)} + \rho_{x_2} \rho_{(x_3)} + \dots \\ + \rho_{x_{r-1}} \rho_{(x_r)} \dots \quad (30)$$

If the dyadic $a, b_{(i)}$ be incomplete the equation (27) will have zero roots.

If we replace l^s by the dyadic $a, b_{(i)}^s$ in (27) we get the Hamilton-Cayley equation for n -space.

6. Binomial dyadics

The dyadic $a, a_{(i)} + l a, b_{(i)}$, where l is a scalar is a binomial dyadic. A binomial dyadic for which

$$l = - \frac{\left\{ \begin{smallmatrix} \rho \\ \rho \end{smallmatrix} \right\}}{\left\{ \begin{smallmatrix} a_h \\ \rho \end{smallmatrix} \right\} \left\{ \begin{smallmatrix} b^{(h)} \\ \rho \end{smallmatrix} \right\}} \quad \dots \quad (1)$$

transforms the vector ρ into one at right angles to it.

The consequent of the product of s binomial dyadics may be expressed in the form

$$[a_{(i)} + l b_{(i)}]^s = a_{(i)} + \binom{s}{1} l b_{(i)} + \binom{s}{2} l^2 b_{(i)}^2 + \dots + l^s b_{(i)}^s \quad \dots \quad (32)$$

which is analogous to the binomial theorem in Algebra.

Assuming (32) true for negative integral values of s , let us put $s = -1$ and define

$$[a_{(i)} + l b_{(i)}]^{-1} = a_{(i)} - l b_{(i)} + l^2 b_{(i)}^2 - \dots + (-1)^r l^r b_{(i)}^r + \dots \quad (33)$$

which forms the co-vector system allied to the system of vectors $a, + l b_i^{-1}$.

The dyadic conjugate to $a, a_{(i)} + l a, b_{(i)}$ is given by

$$a_{(i)} a, + l a_{(i)} b_i^{-1}. \quad \dots \quad (34)$$

The consequent of the product of s such conjugate dyadics may be expressed in the form

$$[a_{(i)} + l b_i^{-1}]^s = a_{(i)} + \binom{s}{1} l b_i^{-1} + \binom{s}{2} l^2 b_i^{-2} + \dots + l^s b_i^{-s}. \quad \dots \quad (35)$$

It may be observed that the co-vector system allied to (35) is expressed in the form

$$[a_{(i)} + lb_{(i)}]^{-s}.$$

It must follow from Hamilton-Cayley equation that a set of only n vectors of type $b_{(i)}$ or b_{-i} in the above formulae are linearly independent.

For a more general binomial dyadic $a_i b_{(i)} + la_i c_{(i)}$ the corresponding powers are more complicated. It may be observed that the co-vector system allied to $b_{(i)} + lc_{(i)}$ is expressible in the form of a series

$$b_h - lb_{i_1} \left\{ \begin{matrix} c(i_1) \\ b_h \end{matrix} \right\} + l^2 b_{i_2} \left\{ \begin{matrix} c(i_2) \\ b_{i_1} \end{matrix} \right\} \left\{ \begin{matrix} c(i_1) \\ b_h \end{matrix} \right\} - \text{etc.} \quad \dots \quad (36)$$

provided it is convergent.

7. Multidyadics

A set of r vectors in n -space defines a multivector of r th order. Denoting the vectors of the set by $\rho_1, \rho_2 \dots \rho_r$ the $\binom{n}{r}$ co-ordinates of the multivector referred to the basis a_i are given by the scalar determinants

$$\left\{ \begin{matrix} a(i_1) & a(i_2) & \dots & a(i_r) \\ \rho_1 & \rho_2 & \dots & \rho_r \end{matrix} \right\}, \quad \dots \quad (37)$$

where $i_1, i_2, \dots, i_r$ is an r -combination in natural order of the numbers $1, 2, \dots, n$.

To consider the linear transformation of a multivector let us first take a bivector $\rho_1 \rho_2$. The double dyadic^s (of type 1) transforming the bivector will be represented in the form

$$\frac{1}{2} \sum_i a_i a_h (b'_{(i)} b'_{(h)} + b''_{(i)} b'_{(h)}), \quad \dots \quad (38)$$

where the summation extends over $\binom{n}{2}$ combinations i, h in natural order of the numbers $1, 2, \dots, n$ taken two at a time. In what follows we shall omit the summation sign assuming this convention for a repeated pair of suffixes.

The transformation of the bivector $\rho_1 \rho_2$ by (38) is represented by means of

$$\frac{1}{2} a_i a_h \left(\left\{ \begin{matrix} b'_{(i)} & b'_{(h)} \\ \rho_1 & \rho_2 \end{matrix} \right\} + \left\{ \begin{matrix} b''_{(i)} & b'_{(h)} \\ \rho_1 & \rho_2 \end{matrix} \right\} \right) \quad \dots \quad (39)$$

which is equivalent to

$$a_i a_h \begin{Bmatrix} a_{(i)} & a_{(h)} \\ \rho'_1 & \rho'_2 \end{Bmatrix}$$

if $\rho'_1 \rho'_2$ denotes the transformed bivector.

It is convenient to express the sum of two terms in the consequent of the double dyadic (38) in the symbolic form

$$\frac{1}{2} \begin{vmatrix} b'_{(i)} & b''_{(i)} \\ b'_{(h)} & b''_{(h)} \end{vmatrix}^* \quad \dots \quad (40)$$

the asterisk distinguishing it from the notation for a determinant.

To express (39) we then adopt the notation

$$\frac{1}{2} a_i a_h \begin{Bmatrix} \begin{vmatrix} b'_{(i)} & b''_{(i)} \\ b'_{(h)} & b''_{(h)} \end{vmatrix}^* \\ \rho_1 \quad \rho_2 \end{Bmatrix} \text{ or } \frac{1}{2} a_i a_h \begin{Bmatrix} \rho_1 \quad \rho_2 \\ \begin{vmatrix} b'_{(i)} & b''_{(i)} \\ b'_{(h)} & b''_{(h)} \end{vmatrix}^* \end{Bmatrix} \quad \dots \quad (41)$$

As for the *resolved form* of the double dyadic (38) we have then the notation

$$\frac{1}{2} a_i a_h \begin{Bmatrix} \begin{vmatrix} b'_{(i)} & b''_{(i)} \\ b'_{(h)} & b''_{(h)} \end{vmatrix}^* \\ \begin{Bmatrix} a_{i_1} & a_{h_1} \end{Bmatrix} \end{Bmatrix} = a_{(i_1)} a_{(h_1)} \quad \dots \quad (42)$$

The above may be represented in the *frame* of a compound matrix of order $\binom{n}{2}$, the element in (ih) th row and $(i_1 h_1)$ th column being

$$\frac{1}{2} \begin{Bmatrix} \begin{vmatrix} b'_{(i)} & b''_{(i)} \\ b'_{(h)} & b''_{(h)} \end{vmatrix}^* \\ \begin{Bmatrix} a_{i_1} & a_{h_1} \end{Bmatrix} \end{Bmatrix} \quad \dots \quad (43)$$

The determinant of order $\binom{n}{2}$ of this matrix will be called the determinant of the double dyadic.

It may be verified that (43) is equivalent to

$$\frac{1}{2} \begin{Bmatrix} \begin{Bmatrix} a_{(i)} & a_{(h)} \\ b'^{-1}_{i_1} & b''^{-1}_{i_1} \end{Bmatrix} \\ \begin{Bmatrix} b'^{-1}_{h_1} & b''^{-1}_{h_1} \end{Bmatrix} \end{Bmatrix}^* \quad \dots \quad (44)$$

In the matrix representation of the product of two double dyadics of type (38) the element in (ih) th row and $(i_2 h_2)$ th column will be of the form

$$\frac{1}{4} \left\{ \begin{array}{cc} b'_{(i)} & b''_{(i)} \\ b'_{(h)} & b''_{(h)} \end{array} \right\}^* \left\{ \begin{array}{cc} c'_{(i_1)} & c''_{(i_1)} \\ c'_{(h_1)} & c''_{(h_1)} \end{array} \right\}^* \dots \quad (45)$$

$$\left\{ \begin{array}{cc} a_{i_1} & a_{h_1} \end{array} \right\} \left\{ \begin{array}{cc} a_{i_2} & a_{h_2} \end{array} \right\}$$

which by means of (44) reduces to the sum of 4 scalar determinants expressed as

$$\frac{1}{4} \left[\begin{array}{cc} b'_{(i)} & b''_{(h)} \\ c'^{-1}_{i_2} & c''^{-1}_{h_2} \end{array} \right] + \begin{array}{cc} b'_{(i)} & b''_{(h)} \\ c''^{-1}_{i_2} & c'^{-1}_{h_2} \end{array} + \begin{array}{cc} b''_{(i)} & b'_{(h)} \\ c'^{-1}_{i_2} & c''^{-1}_{h_2} \end{array} + \begin{array}{cc} b''_{(i)} & b'_{(h)} \\ c''^{-1}_{i_2} & c'^{-1}_{h_2} \end{array} \right]$$

Since

$$\begin{array}{c} \left\{ \begin{array}{c} b'_{(i)} \\ c'^{-1}_{i_2} \end{array} \right\} = \left\{ \begin{array}{c} b'_{(i)} \\ a_h \end{array} \right\} \left\{ \begin{array}{c} c'_{(h)} \\ a_{i_2} \end{array} \right\} = \left\{ \begin{array}{c} (c'b')_{(i)} \\ a_{i_2} \end{array} \right\},$$

the above, on further reduction, may be presented in the form

$$\frac{1}{4} \left[\begin{array}{cc} (c'b')_{(i)} & (c'b'')_{(i)} \\ (c'b')_{(h)} & (c'b'')_{(h)} \end{array} \right]^* \left[\begin{array}{cc} (c''b')_{(i)} & (c''b'')_{(i)} \\ (c''b')_{(h)} & (c''b'')_{(h)} \end{array} \right]^* \dots \quad (46)$$

$$\left\{ \begin{array}{cc} a_{i_2} & a_{h_2} \end{array} \right\} \left\{ \begin{array}{cc} a_{i_2} & a_{h_2} \end{array} \right\}$$

To define the *reciprocal* of the double dyadic (38) we have to consider the determinant of the double dyadic. If $(P_{i_1 h_1}^{i_2 h_2})$ denote the cofactor of the element (43) in the determinant which we represent by R , then the product

$$\frac{1}{2} a_i a_h \left\{ \begin{array}{cc} b'_{(i)} & b''_{(i)} \\ b'_{(h)} & b''_{(h)} \end{array} \right\}^* \frac{(P_{i_1 h_1}^{i_2 h_2})}{R} a_{(i_2)} a_{(h_2)} \dots \quad (47)$$

$$\left\{ \begin{array}{cc} a_{i_1} & a_{h_1} \end{array} \right\}$$

is equivalent to the identical double dyadic $a_i a_h a_{(i)} a_{(h)}$. The resolved form of the reciprocal double dyadic is, therefore, given by

$$a_{i_1} a_{h_1} \frac{(P_{i_1 h_1}^{i_2 h_2})}{R} a_{(i_2)} a_{(h_2)} \dots \quad (48)$$

Following (40), the consequent of a multidyadic of order r involving the sum of $r!$ terms will be represented in the symbolic form

$$\frac{1}{r!} \begin{vmatrix} b'(i_1) & b''(i_1) & \dots & b^{(r)}(i_1) \\ b'(i_2) & b''(i_2) & \dots & b^{(r)}(i_2) \\ \dots & \dots & \dots & \dots \\ b'(i_r) & b''(i_r) & \dots & b^{(r)}(i_r) \end{vmatrix} \quad \dots \quad (49)$$

Calling it a *compound consequent* we note the following elementary properties of it.

(i) Interchange of any two rows changes the sign of a compound consequent.

(ii) Interchange of columns does not alter its explicit form.

(iii) If any constituent be the sum of two others, the compound consequent can be resolved into the sum of two others.

(iv) If b 's fall into s groups of $j_1, j_2, \dots, j_s$ members, respectively, the members in each group be equal to one another, then the number of terms in the compound consequent will be reduced to

$$\frac{r!}{j_1! j_2! \dots j_s!} \quad (\Sigma j_s = r).$$

(v) Interchange of rows and columns is not permitted in a compound consequent.

The *resolved form* of a multidyadic of order r is expressible in the form

$$\frac{1}{r!} a_{i_1} a_{i_2} \dots a_{i_r} \begin{vmatrix} b'(i_1) & b''(i_1) & \dots & b^{(r)}(i_1) \\ b'(i_2) & b''(i_2) & \dots & b^{(r)}(i_2) \\ \dots & \dots & \dots & \dots \\ b'(i_r) & b''(i_r) & \dots & b^{(r)}(i_r) \end{vmatrix} a_{(h_1)} a_{(h_2)} \dots a_{(h_r)} \quad \dots \quad (50)$$

$$\{ a_{h_1} \quad a_{h_2} \quad \dots a_{h_r} \}$$

which like (42) may be represented in the *frame* of a compound matrix of order (r) .

The reciprocal of a multidyadic and the product of a number of multidyadics may be defined as before.

The generalized formula corresponding to (46), for instance, will be

$$\frac{1}{(r!)^2} \sum_c \begin{vmatrix} (c'b')_{(i_1)} & (c'b'')_{(i_1)} & \dots & (c'b^{(r)})_{(i_1)} \\ (c''b')_{(i_2)} & (c''b'')_{(i_2)} & \dots & (c''b^{(r)})_{(i_2)} \\ \dots & \dots & \dots & \dots \\ (c^{(r)}b')_{(i_r)} & (c^{(r)}b'')_{(i_r)} & \dots & (c^{(r)}b^{(r)})_{(i_r)} \end{vmatrix}^* \dots \quad (51)$$

$$\left\{ \begin{matrix} a_{h_1} & a_{h_2} & \dots & a_{h_r} \end{matrix} \right\}$$

where $\sum_c$ denotes the sum of $r!$ elements as one above obtained by permuting the superscripts of c in it in all ways.

References

- ¹ 'Elements of vector analysis' 1884, *Collected works*, Vol. II.
- ² 'An extension of Gibbs' vector analysis to N-space.' *Journal of Mathematics and Physics*, Vol. 15 (1936), pp. 191-204.
- ³ *Messenger of Mathematics*, Vol. XLIX (1920), pp. 167-175.
- ⁴ The usual bold-faced type is not adhered to. All Greek letters and the first few letters of English alphabet denote vectors.
- ⁵ For the elementary properties of it see my paper 'on a class of determinants having geometrical applications,' *This Bulletin*, Vol. 28, pp. 1-12.
- ⁶ Usually called 'the system of reciprocal vectors.' Elsewhere, *This Bulletin* Vol. XXXII, page 52), has been used the term 'conjugate' for 'reciprocal.'
- ⁷ Operating on a vector they define its covariant and contravariant components.
- ⁸ Herein two dyadics $a, b'_{(i)}$ and $a_k b''_{(h)}$ are involved and this may be regarded as the outer product of the two dyadics. Evidently this product is commutative.

THE ENUMERATION OF THE LATIN RECTANGLE OF DEPTH THREE BY MEANS OF A DIFFERENCE EQUATION

By

S. M. KERAWALA

(Communicated by the Secretary)

1. *Introduction*: MacMahon¹ first gave an operational formula for the number of permutations of any number of non-clashing rows, each row consisting of the same totality of n letters, each letter differing from all the rest. Jacob² has supplied reduction formulae for three non-clashing rows, but he remarks that he could not obtain one single recurrence relation for the enumeration. Besides, Jacob's fundamental recurrence formula contains an error, which has vitiated the tables given at the end of his work and deflected him from probably a correct conjecture as to the limit of the ratio of the enumeration to $n!^2$. In this paper I follow closely the method of Jacob, derive a difference equation for the enumeration which Jacob did not, compile fresh tables for values of n up to 15 and justify the conjecture which Jacob abandoned.

2. μ_n will denote the enumeration of the Latin rectangle of depth three. The nature of letters in μ_n is as shown below:

$$\begin{array}{lll} A_1, & A_2, & A_3, \dots, A_n \\ B_1, & B_2, & B_3, \dots, B_n \\ C_1, & C_2, & C_3, \dots, C_n. \end{array}$$

For a permutation to be non-clashing, any one letter of A can appear with any one of $(n-1)$ letters of B and any one of $(n-2)$ letters of C. We have, therefore, that

$$\mu_n = (n-1)(n-2)\alpha_{(n-1)}, \quad \dots \quad (1)$$

where α_n is the enumeration of the non-clashing permutations with the letters

$$\begin{array}{lll} A_1, & A_2, & A_3, \dots, A_n \\ B_0, & B_2, & B_3, \dots, B_n \\ C_0, & C_1, & C_3, \dots, C_n \end{array}$$

the letters in different columns being identical only if their numerical suffixes are identical. Further β_n will denote the non-clashing enumerations with the distribution

$$\begin{array}{ccccccc} A_1, & A_2, & A_3, & A_4, & \dots & A_n \\ B_0, & B_2, & B_3, & B_4, & \dots & B_n \\ C_1, & C_{(n+1)}, & C_3, & C_4, & \dots & C_n \end{array}$$

γ_n the non-clashing enumerations with the distribution

$$\begin{array}{ccccccc} A_1, & A_2, & A_3, & A_4, & \dots & A_n \\ B_1, & B_2, & B_3, & B_4, & \dots & B_n \\ C_0, & C_{(n+1)}, & C_3, & C_4, & \dots & C_n \end{array}$$

δ_n the non-clashing enumerations with the distribution

$$\begin{array}{ccccccc} A_1, & A_2, & A_3, & A_4, & \dots & A_n \\ B_1, & B_2, & B_3, & B_4, & \dots & B_n \\ C_0, & C_2, & C_3, & C_4, & \dots & C_n \end{array}$$

and finally ϵ_n the non-clashing enumerations with the distribution

$$\begin{array}{ccccccc} A_1, & A_2, & A_3, & A_4, & \dots & A_n \\ B_0, & B_2, & B_3, & B_4, & \dots & B_n \\ C_{(n+1)}, & C_2, & C_3, & C_4, & \dots & C_n. \end{array}$$

Considering now the value of α_n , obviously $\alpha_n = \beta_n -$ (all the cases in which B_0 and $C_{(n+1)}$ appear together in β_n).

If $B_0, C_{(n+1)}$ and A_1 turn up together, $\delta_{(n-1)}$ cases are possible; if $B_0, C_{(n+1)}$ and A_2 turn up together, $\delta_{(n-1)}$ more cases result; and if $B_0, C_{(n+1)}$ and any of the remaining $(n-2)$ letters of A appear together, $(n-2)\alpha_{n-1}$ further cases are obtained. Thus

$$\alpha_n = \beta_n - 2\delta_{(n-1)} - (n-2)\alpha_{(n-1)}. \quad \dots (2)$$

Considering the value of δ_n , it is evident that $\delta_n = \mu_n +$ the number of cases in which A_1 appears with $C_0 +$ the number of cases in which B_1 appears with C_0 . These last are $2(n-1)\delta_{(n-1)}$ in number. Hence

$$\delta_n = \mu_n + 2(n-1)\delta_{(n-1)}. \quad \dots (3)$$

The value of ϵ_n is made up of the following different cases :
when

- (a) $A_1, B_0, C_{(n+1)}$ go together, yielding $\mu_{(n-1)}$ cases.
- (b) No two of $A_1, B_0, C_{(n+1)}$ occur together, giving μ_n cases.

(c) Only two of $A_1, B_0, C_{(n+1)}$ appear together, giving $3(n-1)\delta_{(n-1)}$ cases.

$$\text{Hence } \epsilon_n = \mu_n + \mu_{(n-1)} + 3(n-1)\delta_{(n-1)}. \quad \dots (4)$$

Again, β_n is composed entirely of the following different cases :

(a) $A_1, B_0, C_{(n+1)}$ turn up together, giving $\delta_{(n-1)}$ cases.

(b) A_1, B_0 and any one of the $(n-2)$ C 's other than C_1 and $C_{(n+1)}$ occur together, resulting in $(n-2)\gamma_{(n-1)}$ cases.

(c) C_1, B_0 and any one of the $(n-2)$ A 's other than A_1 and A_2 come together, yielding $(n-2)\beta_{(n-1)}$ cases.

(d) C_1, B_0 and A_2 occur together, giving rise to $\epsilon_{(n-1)}$ cases.

(e) No two of A_1, B_0, C_1 occur together, giving δ_n cases.

$$\text{Thus } \beta_n = \delta_n + \delta_{(n-1)} + \epsilon_{(n-1)} + (n-2)\beta_{(n-1)} + (n-2)\gamma_{(n-1)}. \quad \dots (5)$$

Finally, γ_n is composed of all the non-clashing permutations possible when

(a) C_0, B_1, A_2 come together, giving $\epsilon_{(n-1)}$ cases.

(b) C_0, B_1 and any one of the $(n-2)$ A 's other than A_1 and A_2 appear together, giving $(n-2)\beta_{(n-1)}$ cases.

(c) C_0, A_1 and B_2 appear together, giving $\epsilon_{(n-1)}$ cases.

(d) C_0, A_1 and any one of the $(n-2)$ A 's other than B_1 and B_2 appear together, giving rise to $(n-2)\beta_{(n-1)}$ cases.

(e) No two of A_1, B_1, C_0 come together, giving δ_n cases.

$$\therefore \gamma_n = \delta_n + 2\epsilon_{(n-1)} + 2(n-2)\beta_{(n-1)}. \quad \dots (6)$$

Multiplying (2) by $(n-1)$ and using (1),

$$(n-1)(\beta_n - \alpha_n) = 2(n-1)\delta_{(n-1)} + \mu_n$$

is obtained. With the help of (3), this becomes

$$\delta_n = (n-1)(\beta_n - \alpha_n). \quad \dots (7)$$

Substituting for β_n from (2) in (5), we get

$$\alpha_n = \delta_n - \delta_{(n-1)} + \epsilon_{(n-1)} + (n-2)(\beta_{(n-1)} + \gamma_{(n-1)} - \alpha_{(n-1)}).$$

Substituting for $\delta_{(n-1)}$ from (7), we get

$$\alpha_n = \delta_n + \epsilon_{(n-1)} + (n-2)\gamma_{(n-1)}. \quad \dots (8)$$

I shall consider now only the equations (1), (8), (3), (4), (6) and (7). These are six simultaneous difference equations for the six

unknowns $\alpha_n, \beta_n, \gamma_n, \delta_n, \epsilon_n$ and μ_n . Five of the equations are first order equations and one is a zero order equation. If all the unknowns except μ_n be eliminated we should expect to get a fifth order difference equation.

From (1)

$$\mu_{(n+5)} = (n+4)(n+3)\alpha_{(n+4)},$$

With the help of (8), this becomes

$$\mu_{(n+5)} = (n+4)(n+3)[\delta_{(n+4)} + \epsilon_{(n+3)} + (n+2)\gamma_{(n+3)}].$$

Using (6), this becomes

$$\begin{aligned} \mu_{(n+5)} = (n+4)(n+3)[\delta_{(n+4)} + \epsilon_{(n+3)} + (n+2)\delta_{(n+3)} + 2(n+2)\epsilon_{(n+2)} \\ + 2(n+2)(n+1)\beta_{(n+2)}]. \end{aligned}$$

Using (7), this assumes the form

$$\begin{aligned} \mu_{(n+5)} = (n+4)(n+3)[\delta_{(n+4)} + \epsilon_{(n+3)} + (n+2)\delta_{(n+3)} + 2(n+2)\epsilon_{(n+2)} \\ + 2(n+2)(n+1)\alpha_{(n+2)} + 2(n+2)\delta_{(n+2)}]. \end{aligned}$$

Using (1) and (4), this reduces to

$$\begin{aligned} \mu_{(n+5)} = (n+4)(n+3)[\delta_{(n+4)} + (n+2)\delta_{(n+3)} + 5(n+2)\delta_{(n+2)} \\ + 6(n+1)(n+2)\delta_{(n+1)} + 3\mu_{(n+3)} + (2n+5)\mu_{(n+2)} + 2(n+2)\mu_{(n+1)}]. \end{aligned} \quad \dots (9)$$

If now in (9) the μ 's be replaced by δ 's with the help of (3), we have on simplifying

$$\begin{aligned} \delta_{(n+5)} = (n+5)(n+4)\delta_{(n+4)} + (n+5)(n+4)(n+3)\delta_{(n+3)} + (n+4)(n+3)^2\delta_{(n+2)} \\ + 2(n+4)(n+3)(n^2+3n+3)\delta_{(n+1)} - 4(n+4)(n+3)(n+2)n\delta_n. \end{aligned} \quad \dots (10)$$

Equation (10) corresponds to equation (1.12) in Jacob's work. There is an error in this result of Jacob. For the last term of Jacob's equation translated into the notation of this paper is

$$-4(n+4)(n+3)(n+2)(n+1)n\delta_n,$$

whereas the last term of equation (10) is

$$-4(n+4)(n+3)(n+2)n\delta_n.$$

It is this error which has vitiated Jacob's results on pages 336-337.

With the help of (7), (9) becomes

$$\mu_{(n-5)} = (n+3)(n+4) [\mu_{(n+4)} - \mu_{(n+3)} - (n+1)\mu_{(n+2)} + 2(n+2)\mu_{(n+1)}] \\ + 3(n+4)^2(n+3)\delta_{(n+3)}.$$

We have thus the value of δ in terms of μ . Substituting these in (3), we have immediately on simplification

$$(n+3)\mu_{(n+5)} = (n+4)(n^2+8n+17)\mu_{(n+4)} + (n+3)(n+4)(n^2+8n+17)\mu_{(n+3)} \\ + (n+3)(n+4)(n^2+8n+13)\mu_{(n+2)} + 2(n+2)(n+3)(n+4) \\ \times (n^2+5n+3)\mu_{(n+1)} - 4(n+1)(n+2)(n+3)(n+4)^2\mu_n. \quad \dots (11)$$

This is, therefore, the fifth order difference equation looked for, which, though quite simple to derive, Jacob could not achieve.

The problem thus reduces to solving the difference equation (11) being given that $\mu_1=0$, $\mu_2=0$, $\mu_3=2$, $\mu_4=24$ and $\mu_5=552$, these latter values being derived from elementary considerations of non-clashing permutations possible when $n=1, 2, 3, 4, 5$. I could not bring the equation (11) to any of the standard types considered by Milne-Thompson in his *Calculus of Finite Differences*. However, I have calculated the values of μ_n for values of n up to 15, and of the allied enumerations for values of n up to 10, and the results are given in Tables I and II.

To find values of $\frac{\mu_n}{n!^2}$ for $n>15$, we substitute $v_n = \frac{\mu_n}{n!^2}$ in (11)

and obtain

$$v_{(n+5)} = \frac{(n+4)(n^2+8n+17)}{(n+3)(n+5)^2} v_{(n+4)} + \frac{(n^2+8n+17)}{(n+4)(n+5)^2} v_{(n+3)} \\ + \frac{(n^2+8n+13)}{(n+3)^2(n+4)(n+5)^2} v_{(n+2)} + \frac{2(n^2+5n+3)}{(n+2)(n+3)^2(n+4)(n+5)^2} v_{(n+1)} \\ - \frac{4}{(n+1)(n+2)(n+3)^2(n+5)^2} v_n. \quad \dots (12)$$

The volumes of v calculated from (12) are given in Table III.

TABLE I

n	μ_n	$(n!)^2$	$\mu_n / (n!)^2$
1	0	1	0
2	0	4	0
3	2	36	.055 555 556
4	24	576	.041 666 667
5	552	14, 400	.038 333 333
6	21, 280	518, 400	.041 049 383
7	1, 073, 760	25, 401, 600	.042 271 353
8	70, 299, 264	1, 625, 702, 400	.043 242 394
9	5, 792, 853, 248	131, 681, 894, 400	.043 991 266
10	587, 159, 944, 704	18, 168, 189, 440, 000	.044 589 269
11	71, 822, 743, 499, 520	1, 593, 350, 922, 240, 000	.045 076 538
12	10, 435, 273, 503, 677, 440	229, 442, 532, 802, 560, 000	.045 480 990
13	1, 776, 780, 700, 509, 416, 448	38, 775, 788, 043, 632, 610, 000	.045 821 911
14	350, 461, 958, 856, 515, 690, 496	7, 600, 054, 456, 551, 997, 410, 000	.046 113 085
15	79, 284, 041, 232, 622, 163, 140, 608	1, 710, 012, 252, 724, 199, 124, 000, 000	.046 364 504

TABLE II

n	α_n	β_n	γ_n	δ_n	ϵ_n
1	—	—	—	0	1
2	1	1	2	0	0
3	4	5	4	2	2
4	46	58	60	86	44
5	1, 064	1, 274	1, 276	840	1, 008
6	35, 792	41, 728	41, 888	29, 680	34, 492
7	1, 673, 792	1, 912, 112	1, 916, 064	1, 429, 920	1, 629, 280
8	103, 443, 808	116, 346, 400	116, 522, 048	90, 318, 144	101, 401, 344
9	8, 154, 999, 232	9, 059, 742, 176	9, 069, 595, 840	7, 237, 943, 552	8, 030, 787, 968
10	798, 030, 489, 328	877, 746, 364, 288	878, 460, 379, 892	717, 442, 928, 640	788, 377, 273, 856
n	α_n	β_n	γ_n	δ_n	ϵ_n

TABLE III

v_{10}	016 584 014
v_{17}	046 777 073
v_{18}	043 948 245
v_{19}	047 101 050
v_{20}	047 238 276
v_{21}	047 362 192
v_{22}	047 474 638
v_{23}	047 577 122
v_{24}	047 670 925
v_{25}	047 757 096

With the help of equation (12) and the tables given above, it can be readily shown that for $n \geq 7$,

$$v_n < v_{(n-1)} \left[1 + \frac{1}{(n-1)(n-2)} \right].$$

It follows that for $s \geq 7$

$$v_n < v_s \prod_{r=s}^{(n-1)} \left\{ 1 + \frac{1}{(r-1)r} \right\}. \quad \dots \quad (13)$$

Again it is possible to show that for $n \geq 19$,

$$v_n > v_{(n-1)} \left[1 + \frac{1}{(n-1)(n-2)} - \frac{2}{(n+1)^4} \right]$$

from which it follows that for every $s \geq 19$

$$v_n > v_s \prod_{r=s}^{(n-1)} \left[1 + \frac{1}{(r-1)r} - \frac{2}{(r+2)^4} \right]. \quad \dots \quad (14)$$

Thus the sequence v_n is monotonic and bounded and must therefore tend to some limit l such that for every $s \geq 19$

$$v_s \prod_{r=s}^{\infty} \left[1 + \frac{1}{(r-1)r} - \frac{2}{(r+2)^4} \right] < l < v_s \prod_{r=s}^{\infty} \left[1 + \frac{1}{(r-1)r} \right].$$

Taking $s=25$, we have from (13)

$$\begin{aligned} \log l &< \log v_{25} + \sum_{r=25}^{\infty} \frac{1}{(r-1)r} - \frac{1}{2} \sum_{r=25}^{\infty} \frac{1}{(r-1)^2 r^2} + \frac{1}{3} \sum_{r=25}^{\infty} \frac{1}{(r-1)^3 r^3} \\ &= \log v_{25} + \frac{6805}{41472} - \frac{\pi^2}{2} + \sum_{r=1}^{24} \frac{3}{r^4} \\ &= \log v_{25} + \frac{6805}{41472} - 122431993 \\ \therefore l &< 0497884. \end{aligned}$$

Again, from (14), we have that

$$\begin{aligned} \log l &> \log v_{25} + \sum_{r=25}^{\infty} \frac{1}{r(r-1)} - \sum_{r=25}^{\infty} \frac{2}{(r+2)^4} - \frac{1}{2} \sum_{r=25}^{\infty} \frac{1}{r^2(r-1)^2} \\ &+ \sum_{r=25}^{\infty} \frac{2}{r(r-1)(r+2)^4} > \log v_{25} + \frac{95}{1152} - \sum_{r=25}^{\infty} \frac{1}{r^2} - \int_{24}^{\infty} \frac{2dx}{(x+2)^4} \\ &+ \int_{25}^{\infty} \frac{2dx}{(x+2)^6} = \log v_{25} + \frac{95}{1152} - 040810664 - \frac{1}{26364} + \frac{2}{71744535} \end{aligned}$$

from which $l \geq 0497865$.

Now the value of $\frac{1}{e^3} = 0497871$.

Thus the values of both l and $\frac{1}{e^3}$ correct to five places of decimals are the same, viz, 04979. It is, therefore, highly probable that the conjecture $l = \frac{1}{e^3}$ which Jacob discarded is correct.

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CYLINDROID AND KINDRED SURFACES

By

HARIDAS BAGCHI

Introduction: The cylindroid has got special importance in Applied Mathematics on account of the position it occupies in the theory of non-coplanar forces. Generally, most of the geometrical and analytical investigations on the subject have for their ultimate objective the study of statical and kinematical properties connected with the surface. It is, however, felt that, though the cylindroid is only a particular member of a general family of surfaces, a study of this important member from the point of view of Pure Mathematics is desirable, and will also be really helpful. The present investigation has been undertaken from this point of view, and not only many properties of this surface not usually known to Applied Mathematicians have been discussed in this paper, but, it is hoped, it contains also other results which have not been noticed before. For the sake of generalisation some digressional matter bearing on arbitrary cubic scrolls, arbitrary right conoids and arbitrary scrolls has been introduced in the body of the paper.

A few results, mentioned in my "*Note on Cyclides*" (Journal of the Indian Mathematical Society: New Series, Vol. IV, No. 3), have been re-stated here for the sake of clarity and ready reference.

SECTION I

(Characteristic Properties of a Cylindroid and of Associated Surfaces)

1. The cylindroid, first thought of by Sir Robert Ball in course of his investigations on the theory of non-coplanar forces, is definable, in Pure Mathematics, as a cubic surface, whose Cartesian equation, referred to a specially chosen set of rectangular axes, can be thrown into the form

$$z(x^2 + y^2) = 2kxy.$$

It is certainly a cubic scroll of the right conoidal type, the *general* equation to the set of generators being

$$y = \lambda x, \quad z = \frac{2k\lambda}{1 + \lambda^2},$$

where λ is a variable parameter. Plainly the z -axis, which cuts the different generators at right angles, is the line of striction. It may be remarked that this property belongs to *every* variety of right conoid.

Manifestly through an *arbitrary* point $P(0, 0, \gamma)$, situated on the axis OZ , there pass two generating lines, whose parameters are the two roots of the quadratic in λ , *viz.*,

$$\gamma\lambda^2 - 2k\lambda + \gamma = 0.$$

The two roots are equal, if and only if, $\gamma = \pm k$. So there must exist two and only two points on the z -axis; *viz.*, $A(0, 0, k)$ and $B(0, 0, -k)$, such that the two generators, belonging to any of them, are coincident. In the succeeding article other properties of these special points A, B will be put in evidence.

2. It is common knowledge that, whenever a cubic surface possesses a double curve, that curve must be a right line and at the same time the surface must be a scroll, if not a cone. Conversely, every cubic scroll must have a double curve in the shape of a right line, (say L), and its Cartesian equation is amenable to the symbolic form¹

$$u_3 + \varepsilon u_2 + v_2 = 0, \quad \dots (1)$$

(where u_3 is a homogeneous cubic function of x, y , and u_2, v_2 are homogeneous quadratic functions of the same two variables), it being distinctly understood that the origin of co-ordinates is taken somewhere on L , and that L itself is taken as the x -axis. It will readily follow that *every cubic scroll must have a binodal line (L) as its only multiple curve, and that the pairs of binodal planes, answering to the different binodes (situated, no doubt, on L) constitute a pencil of planes in involution, whose focal (or self-corresponding) planes are no else than the two uniplanes, attaching respectively to the two unodes.*

The cubic scroll being taken in the standard form (1), it is not at all difficult to verify that the combined equation to the two uniplanes is

$$J \begin{pmatrix} u_2 & v_2 \\ x & y \end{pmatrix} \equiv \begin{vmatrix} \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} \\ \frac{\partial v_2}{\partial x} & \frac{\partial v_2}{\partial y} \end{vmatrix} = 0. \quad \dots (2)$$

In order to apply these results to a cylindroid, we have to set

$$u_3 \equiv 0, \quad u_2 \equiv x^2 + y^2 \text{ and } v_2 \equiv -2kxy,$$

so that the equation (2) simplifies to $x^2 - y^2 = 0$.

There is thus no escape from the conclusion that the two special points A (0, 0, k) and B (0, 0, $-k$)—referred to in Art. 1—are simply the two unodes of the cylindroid, and that their associated uniplanes, viz., $x - y = 0$ and $x + y = 0$, represent the two focal planes of the involutory system, determined by the pairs of binodal planes.

3. In this article we propose to make a short digression on the plane at infinity and the circle at infinity.

Suppose that from a *real* point R (lying on the plane at ∞) two (imaginary) tangents RP, RQ are drawn to the imaginary circle at ∞ . Take any real point O (lying in the finite portion of space) as the origin of rectangular Cartesian axes, and take the line OR as the z -axis ($x = y = 0$). Then the combined equation of the planes ORP and ORQ must be

$$(y + ix)(y - ix) = 0 \text{ i.e. } x^2 + y^2 = 0.$$

Besides, the plane OPQ is given by $z = 0$.

Thus the combined equation to the trio of planes OQR, ORP, OPQ is

$$z(x^2 + y^2) = 0.$$

Conversely, any Cartesian equation like this is readily seen to represent an *improper* cubic cone, which has the origin for vertex and intersects the plane at ∞ along three right lines, two of which are (imaginary) tangents to the circle at ∞ and the third is their (real) chord of contact.

4. Let us now utilise the lemma that an algebraic surface of degree n being given in the symbolic Cartesian form,

$$U_n + U_{n-1} + U_{n-2} + \dots + U_2 + U_1 + U_0 = 0, \quad \dots \quad (1)$$

(where U_r stands for a homogeneous quantic of degree r in x, y, z), the equation to the cone (of the n th degree), which has the origin for vertex and passes through the section of Ω by the plane at ∞ , is

$$U_n = 0.$$

Applying this lemma to a cylindroid (II), we forthwith deduce that the equation to the cubic cone, whose vertex is O and base is the section of II made by the plane at ∞ , is

$$z(x^2 + y^2) = 0.$$

If we now refer to Art. 3, we quickly recognise that *the section of a cylindroid made by the plane at ∞ is an improper plane cubic, consisting of two (imaginary) tangents to the circle at ∞ and their (real) chord of contact.*

In the next article we intend to examine the converse of this proposition with a view to framing a *geometrical* definition of a cylindroid.

5. Starting with an arbitrary cubic scroll Ω and following the plan adopted in Art. 2, we may exhibit its Cartesian equation in the symbolic form

$$u_3 + zu_2 + v_2 = 0, \quad \dots (1)$$

where presumably u_3, u_2, v_2 are homogeneous functions of x, y of degrees represented by the associated suffixes, and the line of nodes has been taken as the z -axis.

Then, by the lemma (I) of Art. 4, the cubic cone, whose vertex is O and base is the section of Ω by the plane at ∞ , is given by

$$u_3 + zu_2 = 0. \quad \dots (2)$$

If we now impose the condition that the section of the scroll Ω made by the plane at ∞ shall consist of two tangents to the circle at infinity and their chord of contact, then the equation (2) must, by Art. 4, be substantially the same as

$$z(x^2 + y^2) = 0.$$

For this to be possible, the relevant conditions are that u_3 should vanish *identically* and that u_2 should be a numerical multiple of $x^2 + y^2$. These conditions being propitiated, the equation (1), representing the scroll Ω , boils down to the form

$$z(x^2 + y^2) = ax^2 + 2hxy + by^2, \quad \dots (3)$$

(where $-v_2 \equiv ax^2 + 2hxy + by^2$). By a suitable *rigid* transformation this equation is carried over into

$$\zeta(\xi^2 + \eta^2) = 2k\xi\eta.$$

This being the canonical form of a cylindroid, the truth of the converse proposition, envisaged in Art. 4, is manifest.

Unifying the results of this article and of the foregoing one, we can enunciate the final proposition in the following manner:—

The cylindroid represents the only type of (real) cubic scroll, whose section made by the plane at infinity consists of two (imaginary) tangents to the circle at infinity and their (real) chord of contact.

It is scarcely necessary to add that this may be regarded as a sort of *geometrical* definition of a cylindroid.

In the next article our business will be to lay down another (equivalent) *geometrical* definition of the same surface.

6. The *general* (Cartesian) equation of a cubic right conoid can be put in the form

$$\xi = \frac{a_0 \xi^2 + 2a_1 \xi \eta + a_2 \eta^2}{b_0 \xi^2 + 2b_1 \xi \eta + b_2 \eta^2}. \quad \dots (1)$$

By a suitable *rigid* transformation this equation can be carried over into the form

$$z(ax^2 + 2hxy + by^2) = 2kxy. \quad \dots (2)$$

If we now introduce the condition that the two planes

$$ax^2 + 2hxy + by^2 = 0$$

should each touch the circle at ∞ , we must have

$$a = b, \quad h = 0.$$

No generality is lost if we take the common value of a, b to be unity. On this understanding, (2) becomes

$$z(x^2 + y^2) = 2kxy,$$

shewing that the cubic conoid, restricted in the aforesaid manner, is a cylindroid.

Hence, appealing to Articles 3 and 4, we may sum up the above investigations as under:

The cylindroid represents the only type of (real) right conoid of the third degree, whose section made by the plane at infinity consists of three right lines, two of which are (imaginary) tangents to the circle at infinity.

It is worth while to set up a comparison between this definition of a cylindroid and that in Art. 5. It is remarkable that when a cylindroid is to be defined as above, as a special variety of right conoid of the third degree, the qualifying clause to be introduced is that the section of the surface by the plane at ∞ should consist of three rt. lines, two of which touch the $\odot$ at ∞ . That the third (residual) line is the (real) chord of contact of the first two lines is to be looked upon as a *subsidiary property*, which follows by law of causation from the initial defining attributes. In striking contrast to this, there stands out the restrictive condition in the definition in Art. 5, viz., that the section of the cubic scroll by the plane at ∞ should be composed of two tangents to the $\odot$ at ∞ and their chord of contact.

7. As observed in Art. 1, any right conoid—not excepting a cylindroid—partakes of the interesting property of having a *rectilinear* line of striction.

We shall now take cognisance of the most comprehensive form of ruled surface, that can claim a *rectilinear* line of striction.

If for once we assume the lemma² that, if the line of striction of a scroll behaves like a geodesic, it must cut the different generators at a *constant* angle, we immediately deduce that, *when the line of striction of a scroll happens to be a right line, it must intersect the series of generators under a constant angle*. For a right-line, lying wholly on a surface, functions like a geodesic.

An alternative mode of procedure is based on the following (known) proposition:—

If the *general* equations of an arbitrary generator of a skew surface be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \dots (1)$$

(where $\alpha, \beta, \gamma, l, m, n$ are given functions of a single variable parameter t and $l^2 + m^2 + n^2 = 1$), the parametric equations of the line of striction can be written as

$$x = \alpha + lr, y = \beta + mr, z = \gamma + nr, \quad \dots (2)$$

where r is a function of t , defined by

$$r = \frac{\begin{vmatrix} m, n \\ m', n' \end{vmatrix} \times \begin{vmatrix} \beta', \gamma' \\ m, n \end{vmatrix} + \begin{vmatrix} n, l \\ n', l' \end{vmatrix} \times \begin{vmatrix} \gamma', \alpha' \\ n, l \end{vmatrix} + \begin{vmatrix} l, m \\ l', m' \end{vmatrix} \times \begin{vmatrix} \alpha', \beta' \\ l, m \end{vmatrix}}{\begin{vmatrix} m, n \\ m', n' \end{vmatrix}^2 + \begin{vmatrix} n, l \\ n', l' \end{vmatrix}^2 + \begin{vmatrix} l, m \\ l', m' \end{vmatrix}^2} \quad \dots \quad (3)$$

and the dashes placed over the different symbols denote their differential coefficients with respect to t .

In applying this result, we may, at the very start, propose to find all possible skew surfaces that shall have an *assigned* right line (say K) for the line of striction. Without compromising the generality of the problem, we may choose K as the z -axis, so that we have the initial simplification:

$$\alpha=0, \beta=0 \text{ and } \therefore \alpha'=0, \beta'=0.$$

Then (3) reduces to

$$r = \frac{-m\gamma'(mn' - m'n) + l\gamma'(nl' - n'l)}{\Sigma(mn' - m'n)^2}, \quad \dots \quad (4)$$

and (2) becomes

$$x=lr, y=mr, z=\gamma+nr. \quad \dots \quad (5)$$

Since the curve, defined by (5), is none other than the line ($x=y=0$), it is manifest that $r=0$ for all values of t . Hence (4) gives

$$-m(mn' - m'n) + l(nl' - n'l) = 0, \quad (\text{for } \gamma' \neq 0)^3$$

$$\text{i.e.,} \quad -n'(l^2 + m^2) + n(ll' + mm') = 0,$$

$$\text{i.e.,} \quad n'(1 - n^2) + n.nn' = 0. \quad (\because l^2 + m^2 + n^2 = 1 \text{ and } ll' + mm' + nn' = 0),$$

$$\text{i.e.,} \quad n' = 0, \quad \text{so that } n \text{ is a const.}$$

Interpreted geometrically, this corroborates the result that an *arbitrary* generator of the ruled surface makes a *constant* angle with the line of striction ($x=y=0$).

The general equations of a generator, viz. (1), now become

$$\frac{x}{l} = \frac{y}{m} = \frac{z-\gamma}{n}, \quad \dots (6)$$

where n is a constant, and γ, l, m are functions of t , harmonising with the relation $l^2 + m^2 = 1 - n^2$.

It is now easy to see that the Cartesian equation of the surface, obtainable by elimination of t from (6), must assume the symbolic form

$$z - k \sqrt{x^2 + y^2} = f\left(\frac{y}{x}\right), \quad \dots (7)$$

where f is a certain function. This function f being supposed to be arbitrary (and $\therefore$ disposable), the Cartesian equation (7) may be looked upon as representative of the most unrestricted form of scroll that can have an assigned rectilinear line of striction (conveniently chosen as the z -axis). It goes without saying that (7) defines a right conoid and nothing else when the constant k vanishes. So (7) comprises a class of surfaces far wider than that of right conoids.

SECTION II

(Polar Quadrics and Associated Loci)

8. Suppose that Π is an algebraic surface of degree n , whose equation, involving homogeneous co-ordinates x, y, z, w referred to a given tetrahedron, is

$$\phi(x, y, z, w) = 0. \quad \dots (1)$$

Then, as is well known, an arbitrary point $P(x', y', z', w')$ has, associated with it, $n-1$ polar surfaces of the type

$$\Delta^r \phi = 0, \quad \dots (2)$$

where

$$\Delta \equiv x' \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + z' \frac{\partial}{\partial z} + w' \frac{\partial}{\partial w}$$

$$\phi \equiv \phi(x, y, z, w)$$

and the positive integer r ranges from 1 to $n-1$.

Plainly (2) is equivalent to

$$D^{n-1}\overline{\phi}=0,$$

where

$$D \equiv x \frac{\partial}{\partial x'} + y \frac{\partial}{\partial y'} + z \frac{\partial}{\partial z'} + w \frac{\partial}{\partial w'}, \text{ and } \overline{\phi} \equiv \phi(x', y', z', w').$$

In particular, the polar quadric of P (x', y', z', w') with respect to Π is given by

$$D^2.\phi(x', y', z', w')=0,$$

which, written at full length, may be exhibited in the symbolic form:

$$ax^2 + by^2 + cz^2 + dw^2 + 2fyz + 2gzx + 2hxy + 2lxw + 2myw + 2nzw = 0, \dots (3)$$

where $a, b, c, d, f, g, h, l, m, n$ stand respectively for the partial differential co-efficients

$$\begin{aligned} & \frac{\partial^2 \overline{\phi}}{\partial x'^2}, \frac{\partial^2 \overline{\phi}}{\partial y'^2}, \frac{\partial^2 \overline{\phi}}{\partial z'^2}, \frac{\partial^2 \overline{\phi}}{\partial w'^2}, \frac{\partial^2 \overline{\phi}}{\partial y' \partial z'}, \frac{\partial^2 \overline{\phi}}{\partial z' \partial x'}, \frac{\partial^2 \overline{\phi}}{\partial x' \partial y'}, \\ & \frac{\partial^2 \overline{\phi}}{\partial x' \partial w'}, \frac{\partial^2 \overline{\phi}}{\partial y' \partial w'}, \frac{\partial^2 \overline{\phi}}{\partial z' \partial w'}, \end{aligned}$$

and $\overline{\phi}$ stands, as before, for $\phi(x', y', z', w')$.

It is in the fitness of things that the point P traces out a surface-locus or a curve-locus according as its associated polar quadric K is subjected to one or two imposed conditions.

We shall take up a few illustrations of this principle.

Example I. When K is required to be a cone, the point P is known to move on a surface-locus, Ω_1 , which is called the Hessian. The Hessian, whose equation is given in the well-known determinantal form, is of degree $4(n-2)$, and the locus of the vertices of the conical polar quadrics, answering to different points on it, is another surface, which is called the Steinerian and which coincides with the Hessian, if and only if, the original surface Π is a cubic.

Example II. In order to investigate the locus of a point, whose polar quadric is a paraboloid or a surface of revolution, it is expedient—but not obligatory—to use homogeneous co ordinates (x, y, z, w) , referred to that tetrahedron, of which three faces are identical with the three rectangular (Cartesian) co-ordinate planes,

and the fourth face is the plane at infinity. So we may suppose x, y, z to be the same as the ordinary Cartesian co-ordinates, and w to be equal to unity.

On this understanding, it is easy to see that the polar quadric K , as given by (3), becomes a paraboloid, if

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = 0. \quad \dots (5)$$

Hence the locus of a point, whose polar quadric is a paraboloid, is a surface (Ω_2) of degree $3(n-2)$.

Example III. The homogeneous co-ordinates being chosen as in Ex. II, it is abundantly clear that the polar quadric K will be one of revolution, if any one of the following sets of conditions holds:—

$$(i) \quad a - \frac{g^2}{f} = b - \frac{hf}{g} = c - \frac{fg}{h}; \quad \dots (6)$$

$$(ii) \quad g = h = 0, (a-b)(a-c) = f^2; \quad \dots (7)$$

$$(iii) \quad h = f = 0, (b-c)(b-a) = g^2; \quad \dots (8)$$

$$(iv) \quad f = g = 0, (c-a)(c-b) = h^2. \quad \dots (9)$$

Hence, if the polar quadric K be a surface of revolution, the related point P must *either* lie on the curve Σ given by (6) or else coincide with one of the finite number of points defined by each of the three triads of equations (7), (8), (9).

Example IV. Arguing from first principles, we can legitimately expect that, when Π is an algebraic surface of the *ordinary* type, there must exist only a *limited* number of points, whose polar quadrics are right circular cones, and that such points are simply the points of intersection of the curve Σ with the Hessian Ω_1 . It is not inconceivable that, for *exceptional* types of surfaces, points of this description should be infinite in number and claim a curve-locus. Plainly such a contingency can arise, when and only when, the curve Σ lies on the Hessian Ω_1 . That this is, as matter of fact, the case with a cylindroid, will be substantiated in the following article.

Example V. A similar train of arguments makes it plain that *ordinarily* there must exist a *finite* number of points, whose polar quadrics are paraboloids of revolution. Evidently, points such as these are the points of intersection of the curve Σ with the surface Ω_2 .

9. As a preliminary to the application of the results of Art. 8 to a cylindroid Π , we have to represent the position of a point by means of *homogeneous co-ordinates* (x, y, z, w) with reference to the special tetrahedron, three of whose faces are the three Cartesian co-ordinate planes, the fourth face being the plane at ∞ . So, as observed in Art. 8 (Ex. II), x, y, z are identical with the three Cartesian co-ordinates and w may be put equal to unity. With this proviso the homogeneous equation to Π is expressible in the form

$$\phi(x, y, z, w) = 0,$$

where $\phi(x, y, z, w) \equiv z(x^2 + y^2) - 2kxyw = 0$.

It may be noted, for future use, that the polar quadric K of an arbitrary point $P(x', y', z', w')$ is given by the homogeneous equation

$$x' \frac{\partial \phi}{\partial x} + y' \frac{\partial \phi}{\partial y} + z' \frac{\partial \phi}{\partial z} + w' \frac{\partial \phi}{\partial w} = 0,$$

or by the equivalent Cartesian equation

$$z'.x^2 + z'.y^2 + 2y'.yz + 2z'.zx - 2kxy - 2ky'x - 2kx'y = 0. \quad \dots \quad (A)$$

Example (i). To find the Hessian Ω_1 , we may seek the locus of P , subject to the condition that (A) may denote a cone. Anyway, the Hessian is easily obtained in the simple form

$$(x^2 - y^2)^2 = 0. \quad \dots \quad (1)$$

Recollecting the results of Art. 2, we learn that the Hessian is an improper quartic surface, compounded of the two uniplanes ($x + y = 0$ and $x - y = 0$), each counted twice.

Furthermore, reliance being placed on the known theorem that the parabolic curve of a surface Π is its *partial* curve of intersection with its own Hessian—the residual curve of intersection being possibly the multiple curve or curves (if any) of Π —the legitimate inference is that the parabolic curve of the cylindroid is a degenerate twisted curve of the second degree, consisting of the two rt. lines:

$$\left. \begin{array}{l} y = x, \\ z = k; \end{array} \right\} \text{ and } \left\{ \begin{array}{l} y = -x \\ z = -k. \end{array} \right.$$

Hence by Art. 1 we are led to believe that these two constituent lines of the degenerate parabolic curve can also be designated as the two generators passing respectively through the two unodes $(0, 0, k)$ and $(0, 0, -k)$.

Example (ii). To find the surface-locus Ω_2 of points whose polar quadrics are paraboloids, we have to look to (5) of Art. 8. On the strength of (A) of this art., the equation of Ω_2 can in a trice be simplified to the form

$$z(x^2 + y^2) = -2kxy. \quad \dots (2)$$

Verily, then, Ω_2 is also a cylindroid, which may be regarded as the exact prototype of the original cylindroid and is, so to say, its *image* in the xy -plane.

Example (iii). To determine the curve-locus (Σ) of a point whose polar quadric is a surface of revolution, the needful is to reckon with the equation (6) of Art. 8, which, in the present case, reduces to

$$x^2 = y^2 \text{ and } z + k\frac{x}{y} = \frac{xy}{k}. \quad \dots (3)$$

Hence the curve Σ is degenerate and is composed of two parabolas Σ_1, Σ_2 , defined respectively by the pairs of Cartesian equations

$$\left. \begin{array}{l} y = x \\ x^2 = k(k + z) \end{array} \right\} \text{ and } \left\{ \begin{array}{l} y = -x \\ x^2 = k(k - z) \end{array} \right. \quad \dots (4)$$

Example (iv). When, however, we search for points whose polar quadrics with respect to a cylindroid are right circular cones, we are confronted with an exceptional case, envisaged in Ex. IV of Art. 8. The reason is not far to seek. For one of the two conditions marked (3), which have to be fulfilled by a point P in order that its polar quadric may be one of revolution, is precisely the same as the condition for P to lie on the Hessian Ω_1 . Thus, unlike an ordinary algebraic surface, the cylindroid *does admit* of an infinite number of points, having their polar quadrics in the form of right circular cones. In point of fact, the two parabolas Σ_1, Σ_2 must, when taken together, be regarded as the complete (improper) curve-locus of a point whose polar quadric is a right circular cone. By way of tautology, we may affirm that, if the polar quadric of a point with

respect to cylindroid be a surface of revolution, it must be a right cone and at the same time the point itself must lie on one or other of the two parabolas Σ_1, Σ_2 .

To study more fully the aggregate of right circular and conical polar quadrics, it is incumbent upon us to discriminate between two cases as follows:—

Case I. First suppose that $P(x', y', z')$ lies on the parabola Σ_1 . Then we must have

$$y' = x' \quad \text{and} \quad x'^2 = k(k + z'), \quad \dots (5)$$

so that the polar quadric K , as defined by (A), takes the form

$$z'(x^2 + y^2) + 2x'(yz + zx) - 2kxy - 2kx'(x + y) = 0.$$

By elementary algebraic manipulations, this equation can, by aid of (5), be brought to the form

$$x^2 + y^2 + (z - k)^2 = \frac{1}{x'^2} [k(x + y) - x'z + kx']^2,$$

which, on the face of it, typifies a right circular cone, whose vertex is the unode $(0, 0, k)$ and axis is the line

$$x = y = \frac{z - k}{\frac{x'}{k}}.$$

One cannot, then, get away from the conclusion that the polar quadrics, answering to the ∞^1 of points, lying on the parabola Σ_1 , are so many right circular cones having a common vertex, viz., the unode $(0, 0, k)$. It is remarkable that the locus of the axes of these cones is the associated uniplane ($y = x$). (Art. 2.)

Case II. Secondly suppose that $P(x', y', z')$ lies on the parabola Σ_2 . Then, as a matter of course, we have

$$y' = -x' \quad \text{and} \quad x'^2 = k(k - z'),$$

so that the polar quadric K , as given by (A), assumes the form

$$z'(x^2 + y^2) - 2x'(yz - zx) - 2kxy + 2kx'(x - y) = 0.$$

Elementary algebraic artifice converts this into

$$x^2 + y^2 + (z + k)^2 = \frac{1}{x'^2} [k(x - y) + x'z + kx']^2.$$

This certainly points to the conclusion that K is a right circular cone, whose vertex is the unode $(0, 0, -k)$ and axis is the line

$$x = -y = \frac{z' + k}{\frac{x'}{k}}.$$

Hence we are entitled to conclude that the polar quadrics of the ∞^1 of points, situated on the parabola Σ_2 are a series of right circular cones, whose vertex is a *fixed* point, *viz.*, the unode $(0, 0, -k)$, and whose axes are constrained to lie in a fixed plane, *viz.*, the related uniplane $(y = -x)$. (Art. 2.)

Example (v). When, however, we want a point whose polar quadric shall be a paraboloid of revolution, we have simply to note the several points of intersection of the two parabolas Σ_1, Σ_2 with the image of a cylindroid, *viz.*, Ω_2 (Ex. ii). Without much ado one can demonstrate that the only points, whose polar quadrics are paraboloids of revolution, are the two unodes $(0, 0, k)$ and $(0, 0, -k)$. The (paraboloidal) polar quadrics of these two points are both degenerate and consist of the two uniplanes counted twice, *viz.*, $(x - y)^2 = 0$ and $(x + y)^2 = 0$.

10. It is worth while to seek for analogous properties of a cubic scroll. Take the nodal line of the cubic scroll as the z -axis and throw its Cartesian equation into the symbolic form

$$u_3 + zu_2 + v_2 = 0, \quad \dots \quad (1)$$

where u_3, u_2, v_2 are homogeneous functions of x, y of degrees defined by the respective suffixes.

Homogeneous co-ordinates being now adopted on the plan of Art. 8 (Examples ii, iii, iv, v), the *homogeneous equation*, equivalent to (1), becomes

$$\phi(x, y, z, w) = 0,$$

where $\phi(x, y, z, w) \equiv u_3 + zu_2 + wv_2$.

The equation of the Hessian takes the form

$$\left\{ J \begin{pmatrix} u_2 & v_2 \\ x & y \end{pmatrix} \right\}^2 = 0.$$

Since, by Art. 2, the Jacobian equation

$$J \begin{pmatrix} u_2 & v_2 \\ x & y \end{pmatrix} = 0 \quad \dots (2)$$

denotes the two uniplanes of the given scroll, we can conclude that the Hessian (quartic) of any given cubic scroll—be it a cylindroid or not—is composed of the two uniplanes, each reckoned twice.

The determination of the parabolic curve of the scroll calls for a scrutiny of its curve of intersection with the degenerate Hessian (2). So the preliminary step is to solve the binary quadratic (2) for the ratio $y:x$, so as to express the individual equations of the two uniplanes in the forms

$$y = \lambda x \quad \text{and} \quad y = \mu x,$$

where λ and μ are determinate scalars.

The equation (1) of the given scroll may now be taken in the form

$$f_3(x, y) + \lambda f_2(x, y) + \psi_2(x, y) = 0, \quad \dots (3)$$

provided that we set

$$u_3 \equiv f_3(x, y), \quad u_2 \equiv f_2(x, y) \quad \text{and} \quad v_2 = \psi_2(x, y).$$

It is now a pleasant job to verify that the complete intersection of the scroll (1) or (3) with the uniplane $y = \lambda x$ consists of the nodal line ($x = y = 0$), counted twice, together with the right line defined by

$$y = \lambda x \quad \text{and} \quad \lambda f_3(1, \lambda) + \lambda f_2(1, \lambda) + \psi_2(1, \lambda) = 0. \quad \dots (4)$$

Similarly, the intersection of the scroll with the other uniplane $y = \mu x$ consists of the same nodal line (reckoned twice) and the rt. line defined by

$$y = \mu x \quad \text{and} \quad \mu f_3(1, \mu) + \mu f_2(1, \mu) + \psi_2(1, \mu) = 0. \quad \dots (5)$$

We cannot, therefore, get away from the fact the two lines (4) and (5) constitute the parabolic curve. To characterise these lines adequately we observe in the first place that the *typical* equations of the generating lines of the Scroll (3) are

$$y = tx \quad \text{and} \quad t f_3(1, t) + t f_2(1, t) + \psi_2(1, t) = 0,$$

where t is a variable parameter. Consequently, through an arbitrary point $P(0, 0, \gamma)$, lying on the nodal line, there pass two generators,

whose parameters are given by the quadratic in t , viz.,

$$\gamma f_2(1, t) + \psi_2(1, t) = 0.$$

These two generators will coincide, when and only when, this quadratic equation has two equal roots, necessitating the quadratic expression

$$\gamma f_2(x, y) + \psi_2(x, y) \equiv \gamma u_2 + v_2$$

to become a perfect square. In other words, the two generators passing through P are coincident, if and only if, the two attached biplanes $\gamma u_2 + v_2 = 0$ coincide. So the two unodes are the only points for each of which the associated generators are identical. Moreover, the generators, passing through the two unodes, are respectively identical with the two lines (4) and (5).

The final results may then be condensed thus:

The parabolic (or spinodal) curve of a cubic scroll of the most general form consists of two right lines, which can be designated as the generators, passing through the two unodes.

This is a generalisation of the corresponding theorem on a cylindroid. (See Art. 9, Ex i.)

11. We state the following theorem on the locus of points whose polar quadrics, with respect to a cubic scroll (II), are paraboloids:

The locus of points, whose polar quadrics with respect to a cubic scroll of the unrestricted kind are paraboloids, is another cubic scroll, claiming the self-same nodal line as the first.

Readers may supply the proof and also investigate other kindred loci connected with an arbitrary cubic scroll in a manner analogous to that adopted in the case of a cylindroid (Art. 9.)

12. The cylindroid (II), defined usually by the Cartesian equation

$$z(x^2 + y^2) = 2kxy,$$

can undoubtedly be thrown into the parametric form

$$x = p \cos q, \quad y = p \sin q, \quad z = k \sin 2q, \quad \dots \quad (1)$$

where p, q are two independent parameters.

The formula of rectification in compact form is

$$ds^2 = dp^2 + D^2 dq^2,$$

where

$$D^2 \equiv p^2 + 4k^2 \cos^2 2q.$$

Hence we conclude that the series of parametric curves $q = \text{const.}$ are a family of geodesics and that the other set of parametric curves, viz., $p = \text{const.}$, are the associated set of geodesic parallels.

The parametric curves $q = \text{const.}$ are the generators of the cylindroid. It follows, then, that the curves $p = \text{const.}$ may also be designated as the orthogonal trajectory of the set of generators.

It is clear on all hands that an arbitrary curve (say, Γ_a), belonging to the system $p = a$, is representable by the Cartesian (parametric) equations

$$x = a \cos t, y = a \sin t, z = k \sin 2t, \quad \dots (1)$$

and is $\therefore$ a twisted quartic curve, formed by the intersection of the right circular cylinder

$$x^2 + y^2 = a^2 \quad \dots (2)$$

$$\text{with the paraboloid} \quad a^2 z = 2kxy. \quad \dots (3)$$

Apropos of this we may observe that, through any assigned twisted quartic curve, there can always be drawn at least one quadric but an infinity of cubic surfaces. Further, if the number of quadrics, that pass through a given (twisted) quartic, exceeds unity, it must be infinite. As a matter of fact, this differentiating feature forms the basis of classification of twisted quartic curves.

To be more precise, a twisted quartic (curve) is said to be (i) of the first species, when it is the complete intersection of two (and $\therefore$ of infinitely many) quadrics; and (ii) of the second species, when it is the partial intersection of a quadric and a cubic surface. In this latter case, the residual intersection is, of course, a curve of the second degree.

Furthermore, quartic curves of the second species are, without exception, *unicursal*, but those of the first species are generally *bicursal* but exceptionally *unicursal*. As an illustration of this exceptional variety of *unicursal* quartic curve of the first species, we may mention the assemblage of curves of the type $\{\Gamma_a\}$, defined by (1). In support of this statement, we have simply to introduce a new parameter u , connected with the old one, viz., t , by the relation $u = \tan \frac{t}{2}$, so that the parametric equations (1) of Γ_a may be converted into

$$x = a \cdot \frac{1-u^2}{1+u^2}, y = a \cdot \frac{2u}{1+u^2}, z = 4k \cdot \frac{u(1-u^2)}{(1+u^2)^2}$$

These equations prove the assertion that Γ_a is a unicursal quartic curve of the first species and lies in the *finite* portion of space (so long as a is finite). Summing up, we can say that the *orthogonal* trajectory of the set of generating lines of a cylindroid is composed of a family of unicursal quartic curves of the first species.

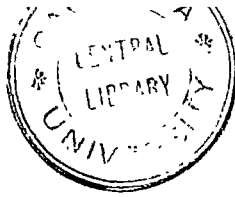
It may be added that the original cylindroid admits of generation as a surface-locus from the ∞^1 of quartic curves $\{\Gamma_a\}$, obtained by varying the constant a .

References

¹ See Salmon's *Geometry of Three Dimensions* (Ed. by Rogers), Vol. II (1915), p. 168, art. 520.

² See Forsyth's *Differential Geometry* (1912), p. 385, and the author's *Course of Geometrical Analysis* (1926), p. 417.

³ For if $\gamma=0$, then $\gamma=\text{const.}$, and an *arbitrary* generator would have to pass through a *fixed* pt. viz. $(0, 0, \gamma)$, and the surface would become a cone and not a scroll.



PROPERTIES OF MUTUALLY SELF-POLAR TETRAHEDRA

By

SAHIB RAM MANDAN

(Communicated by Prof. C. V. H. Rao)

The object of this paper is to give an account of mutually self-polar tetrahedra as a generalization of mutually self-polar triangles.¹

I start with the definition and existence, of mutually self-polar tetrahedra, which implies the commutative Law of Multiplication [§§1, 2].

Next I arrive at the desmic configuration by imposing certain conditions on a pair of mutually self-polar tetrahedra [§§3, 4, 7, 8].

Then there arise a number of theorems as follow :

Any two mutually self-polar tetrahedra are reciprocal *w. r. t.* four pairs of quadrics, quadrics in each pair having four generators common [§ 6].

If two mutually desmic tetrahedra be self-polar to another tetrahedron, a pair of skew edges of the latter tetrahedron coincides with a pair of edges of the tetrahedron desmic with the former two [§ 9].

There exist two tetrahedra desmic with two mutually self-polar tetrahedra [§ 11].

There exists a set of five tetrahedra such that every two of them are desmic, their centres of perspectivity being the points common to the remaining three. Two pairs of edges of each tetrahedron of a triad of them are generators of a quadric for which the remaining two are self-conjugate. There are ten such quadrics called fundamental² for Kummer's surface.

1. *Existence.* Let $A_1B_1C_1D_1$ be a tetrahedron T_1 ; B_2, C_2, D_2 be the harmonic inverses³ of a given point A_2 *w. r. t.* the pairs of lines A_1B_1, C_1D_1 ; C_1A_1, B_1D_1 ; A_1D_1, B_1C_1 respectively. Now, if we call $B_2C_2D_2$ as the polar plane of A_2 *w. r. t.* T_1 , we see that this polarity⁴ is the same as referred to the quartic surface consisting of the four faces of T_1 . Since the edges of T_1 are double lines of this surface and the polar planes of B_2, C_2, D_2 *w. r. t.* it pass through

A_2 , the polar cubic⁴ of A_2 w. r. t. it contains B_2, C_2, D_2 and the edges of T_1 . The section of this cubic by the plane $B_2C_2D_2$ is the degenerate curve* consisting of the three lines C_2D_2, B_2D_2, B_2C_2 , as the diagonal triangle of the quadrilateral formed by the lines common to the plane $B_2C_2D_2$ and the faces of T_1 is seen to be $B_2C_2D_2$ and thence the lines C_2D_2, B_2D_2, B_2C_2 meet the pairs of edges of T_1 .

Hence the polar plane of any point on the lines C_2D_2, B_2D_2, B_2C_2 w. r. t. T_1 passes⁴ through A_2 and that of D_2 is seen to be $A_2B_2C_2$. Now, if we take a pair of points B_2', C_2' on B_2C_2 separated harmonically by the lines B_1C_1, A_1D_1 , the tetrahedron $A_2B_2'C_2'D_2$ is such that its faces are the polar planes of the opposite vertices w. r. t. T_1 . Such a tetrahedron may be said to be *selfpolar* to T_1 which is again seen to be related to $A_2B_2'C_2'D_2$ in the same manner. Hence T_1 and $A_2B_2'C_2'D_2$ constitute a pair of *mutually selfpolar tetrahedra*.

2. *Commutative Law of Multiplication.* Let the vertices of a tetrahedron T be denoted symbolically³ by A, B, C, D and A_1 be a point such that $A_1 + A + B + C + D \equiv 0$. Now the polar plane of A_1 w. r. t. T , as defined in §1, is seen to contain six points of the type $B-C$; hence any point B_1 in this plane may be taken as $\theta(A-B) + \phi(B-C) + \psi(C-D)$, i.e., $B_1 \equiv \theta A - (\theta - \phi)B - (\phi - \psi)C - \psi D$. Again the polar plane of B_1 w. r. t. T contains the points $\theta A + (\theta - \phi)B, (\theta - \phi)B - \psi D, (\theta - \phi)B - (\phi - \psi)C$. Hence the polar plane of B_1 w. r. t. T will pass through A_1 if the four points, $A_1, \theta A + (\theta - \phi)B, (\theta - \phi)B - \psi D, (\theta - \phi)B - (\phi - \psi)C$ are linearly related. For this, in view of $-A_1 \equiv A + B + C + D$, we need $(\phi - \psi)[\theta(A + B + C + D) - \{\theta A + (\theta - \phi)B\}] - \theta\{(\phi - \psi)C - (\theta - \phi)B\} \equiv K\{(\theta - \phi)B - \psi D\}$.

We have, on comparing the coefficients of B, C, D respectively,

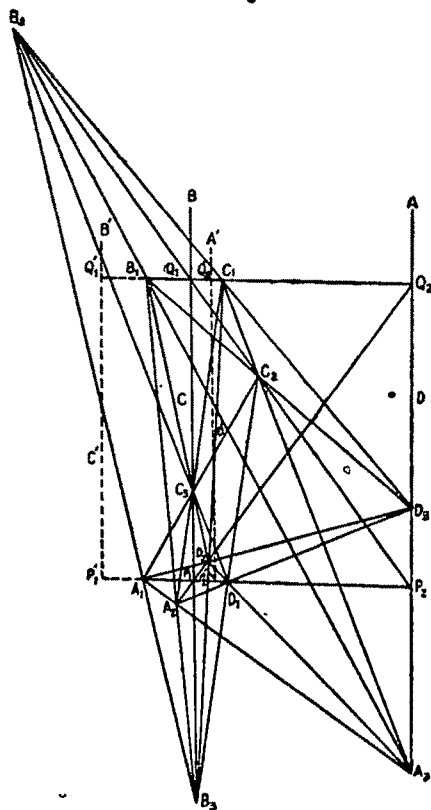
$$(\phi - \psi)\{\theta - (\theta - \phi)\} - \theta\{-(\theta - \phi)\} = K(\theta - \phi),$$

$$(\phi - \psi)\theta - \theta(\phi - \psi) = 0, \quad (\phi - \psi)\theta = K(-\psi).$$

The second of these relations is equivalent to $(\phi - \psi)\theta = \theta(\phi - \psi)$. When this relation holds, i.e., when the *Commutative Law of Multiplication* holds, the first and third relations lead to $\phi(\theta - \psi)(\theta - \phi + \psi) = 0$. The three factors, each equated to zero, give the three forms for the symbol B_1 according as B_1 is on the three lines of §1.

* Mr. Asghar Hameed proves it analytically.

3. If T_1, T_3 be a pair of mutually selfpolar tetrahedra (a pair of skew edges of T_1 then meets a pair of edges of T_3), a second pair of skew edges of T_1 meet a second pair of edges of T_3 , and the third pair of edges of T_1 meet the third pair of edges of T_3 and thence T_1 is desmic with T_3 .



Let A, B, C, D , be the tetrahedron T_1 . As T_1 is selfpolar to T_3 , let A_1D_1, B_1C_1 meet B_3C_3 in P_1, Q_1 and A_3D_3 in P_2, Q_2 respectively such that $(A_1D_1, P_1P_2) = (B_1C_1, Q_1Q_2) = -1 = (B_3C_3, P_1Q_1) = (A_3D_3, P_2Q_2)$. The lines $A_3B_3, C_3D_3, P_1P_2, Q_1Q_2$ then generate a quadric. Suppose further that C_1A_1, B_1D_1 meet C_3A_3, B_3D_3 , then C_1A_1, B_1D_1 are seen to be separated harmonically by the pairs of points C_3, A_3 and B_3, D_3 , hence $A_3B_3, C_3D_3, C_1A_1, B_1D_1$ generate another quadric. The two quadrics have A_3B_3, C_3D_3 as their common generators and therefore they have in common

two other generators which meet $A_3B_3, C_3D_3, C_1A_1, B_1D_1, P_1P_2, Q_1Q_2$; they cannot be then other than A_1B_1, C_1D_1 .

Again let the triads of the planes $B_1C_1B_3C_3, C_1A_1C_3A_3, A_1B_1A_3B_3; A_3B_1C_1D_3, A_1B_3C_1D_3, A_1B_1A_3B_3; A_1B_1C_3D_3, A_3B_1C_1D_3, C_1A_1C_3A_3; A_1B_3C_1D_3, A_1B_1C_3D_3, B_1C_1B_3C_3$ meet in the points A_2, B_2, C_2, D_2 respectively. The six planes $B_1C_1D_3, B_1C_1A_2; C_1A_1D_3, C_1A_1A_2; A_1B_1D_3, A_1B_1A_2$ meet in pairs in three lines lying in a plane. Hence T_i ($i=1, 2, 3$) are the tetrahedra of a desmic configuration and we see that $A_2D_2B_2C_2$ meet $A_1D_1, B_1C_1, A_3D_3, B_3C_3$; therefore they coincide with P_1Q_2, P_2Q_1 .

4. If T_1 is a tetrahedron selfpolar to another tetrahedron T_3 and perspective to it from a point A_2 , T_1 is desmic with T_3 .

Let A, B, C, D be the tetrahedra T_1 and A_1D_1, B_1C_1 meet B_3C_3, A_3D_3 as in §3. As $(B_1C_1, Q_1Q_2) = (B_3C_3, Q_1P_1)$, it follows that A_2 lies on P_1Q_2 if we suppose that $A_1A_3, B_1B_3, C_1C_3, D_1D_3$ meet in A_2 . Now A_1A_3 meets B_1B_3 , therefore A_3B_3 meets A_1B_1 ; again as $(B_1C_1, Q_1Q_2) = (A_1D_1, P_1P_2)$ and A_3B_3 meets A_1B_1, P_1Q_1, P_2Q_2 , it meets C_1D_1 . Similarly C_3D_3 meets C_1D_1, A_1B_1 . Hence by §3, T_1 is desmic with T_3 .

5. Let T, T_1 be a pair of mutually selfpolar tetrahedra, $ABCD, A_1B_1C_1D_1$ such that BC, AD meet A_1D_1 in P_1, P_2 and B_1C_1 in Q_1, Q_2 respectively and $(BC, P_1Q_1) = (AD, P_2Q_2) = -1 = (A_1D_1, P_1P_2) = (B_1C_1, Q_1Q_2)$. The two tetrahedra selfpolar to T, T_1 and perspective to them respectively from a point on either of the lines P_1Q_2, P_2Q_1 are mutually selfpolar.*

Let A_2 be a point on P_1Q_2 . A pair of edges, say B_3C_3, A_3D_3 of the tetrahedron T_3 selfpolar to T_1 and perspective to it from A_2 is seen to be coincident with the pair of lines P_1Q_1, P_2Q_2 such that $(B_3C_3, P_1Q_1) = -1 = (A_3D_3, P_2Q_2)$ while a pair of edges of, say $A_3'D_3', B_3'C_3'$ of the tetrahedron T_3' selfpolar to T and perspective to it from A_2 coincides with the pair of lines P_1P_2, Q_1Q_2 such that $(A_3'D_3', P_1P_2) = -1 = (B_3'C_3', Q_1Q_2)$. Hence T_3 is selfpolar to T_3' . Similar is the case when we start from a point on P_2Q_1 .

6. Any two mutually selfpolar tetrahedra are reciprocal w. r. t. four pairs of quadrics, quadrics in each pair having four generators common.

* This result and those in §§7,12 are obtained by from Mr. Asghar Hameed analytically; here is the geometric treatment.

Let T, T_1 be the tetrahedra as in §5. Now consider the quadrics for which AD, B_1C_1 and BC, A_1D_1 are pairs of polar lines.

Let Σ_1 be one for which AB is polar to A_1B_1 and thence CD is polar to C_1D_1 ; and Σ_1' be another for which AB is polar to C_1D_1 and thence CD is polar to A_1B_1 . As $(A_1D_1, P_1P_2) = (CB_1P_1Q_1)$, CA_1, BD_1 meet in a point P on the line P_2Q_1 . The polar line of CA_1 *w. r. t.* Σ_1 is seen to be the line l common to the planes $C_1A_1D_1$ and ABC ; and that of BD_1 is the line l' common to the planes $A_1B_1D_1$ and BCD . Hence the polar plane of P *w. r. t.* Σ_1 is the plane l, l' which is also seen to be the polar plane of P *w. r. t.* Σ_1' . Let this plane meet P_2Q_1 in the point Q , say. Hence P, Q , and P_2, Q_1 are pairs of conjugate points for both Σ_1, Σ_1' . Therefore PQ meets Σ_1, Σ_1' in the same points say B_2, C_2 which are the united elements of the involution determined by P, Q and P_2, Q_1 as pairs of corresponding points. Hence Σ_1, Σ_1' have the same generators P_1B_2, P_1C_2 at P_1 and Q_2B_2, Q_2C_2 at Q_2 . Σ_1, Σ_1' give us one pair. We have another pair Σ_2, Σ_2' for which AB is polar to C_1A_1, B_1D_1 respectively.

Similarly we have two other pairs of quadrics, for which the given tetrahedra are reciprocal, which have the same generators at P_2, Q_1 .

Analytically the equations of the quadrics may be put down as

$$v(x-t)^2 \pm \{(y+z)^2 \pm 2(y-z)(x+t)\} = 0;$$

and
$$(y-z)^2 \pm v\{(x+t)^2 \pm 2(x-t)(y+2)\} = 0$$

where $(1, 1, 1, 1), (-1, v, -v, 1), (-1, -v, v, 1), (1-1, -1, 1)$ are the co-ordinates of A_1, B_1, C_1, D_1 respectively referred to T .

7. Let T, T_1 be the tetrahedra as in §6 and if we apply harmonic inversion *w. r. t.* a pair of skew generators common to a pair of quadrics for which T, T_1 are reciprocal such that T inverts into T_3' and T_1 into T_3 , then T_3', T_3 are respectively desmic with T, T_1 .

Let us invert T_1 *w. r. t.* P_1B_2, Q_2C_2 [§6]. P_1, Q_2 invert into themselves; P_2 inverts into Q_1 . The inverses of A_1, D_1 are the points C_3, B_3 where P_1Q_1 meets C_2A_1, C_2D_1 . Now A_1B_3, C_3D_1 meet in B_2 for $(A_1D_1, P_1P_2) = (B_3C_3, P_1Q_1) = -1$. The inverses of B_1, C_1 are the points A_3, D_3 where B_1B_2, B_2C_1 meet P_2Q_2 . Hence T_1 is selfpolar to $A_3B_3C_3D_3$, i.e., T_3 and perspective to it from B_2 . T_1 is therefore, by §4, desmic with T_3 .

Similarly for T we find T_3' desmic with it.

8. *The eight vertices of two mutually selfpolar tetrahedra are associated if and only if the tetrahedra are mutually desmic.*

Let the tetrahedra be T_1, T_3 and suppose their vertices are a set of eight associated points, then T_1, T_3 must be self-conjugate for a quadric Σ .⁶

Now if T_1 be $A_1B_1C_1D_1$ and A_1D_1, B_1C_1 meet A_3D_3, B_3C_3 as in §3; and if A_1D_1 meets Σ in P_1', P_2' ; B_1C_1 in Q_1, Q_2' ; A_3D_3 in A, D ; B_3C_3 in B, C ; then P_1Q_1', P_2Q_2' and $P_1'Q_2', P_2'Q_1'$ are pairs of generators of Σ of different systems; therefore either of the pairs, say $P_1'Q_1', P_2'Q_2'$ meets either of the other pairs of generators AB, CD or CA, BD , say AB, CD of Σ .

Again AB, CD, A_1D_1, B_1C_1 are met by $AD, BC, P_1'Q_1', P_2'Q_2'$, hence these sets of lines generate another quadric Σ' of which, by the Lemma,* either A_3B_3, C_3D_3 or C_3A_3, B_3D_3 , say A_3B_3, C_3D_3 and either A_1B_1, C_1D_1 or C_1A_1, B_1D_1 , say A_1B_1, C_1D_1 , are generators, i.e., A_1B_1, C_1D_1 meet A_3B_3, C_3D_3 . Hence, by §3 T_1 is desmic with T_3 .

9. *Let T_i ($i=1, 2, 3$) be the tetrahedra of a desmic configuration; then two of them, say T_1, T_2 may be selfpolar to a tetrahedron T' different from T_3 if and only if a pair of skew edges of T' coincide with a pair of skew edges of T_3 .*

Let T' be $A'B'C'D'$ and T_1 as in §2. As T' is selfpolar to T_1 , let $B'C', A'D'$ meet A_1D_1 in P_1', P_2' and B_1C_1 in Q_1', Q_2' respectively such that $(B'C', P_1'Q_1') = (A'D', P_2'Q_2') = -1 = (A_1D_1, P_1'P_2') = (B_1C_1, Q_1'Q_2')$.

* Lemma : Let P_i ($i=1, 2, 3, 4$) be points on a line l and P_i' on another line l' such that $(P_1P_2P_3P_4) = (P_1'P_2'P_3'P_4')$.

$P_1P_1', P_2P_2', P_3P_3', P_4P_4'$ generate a quadric and $P_1'P_2, P_2'P_1, P_3'P_4, P_4'P_3$ generate another quadric as $(P_1P_2P_3P_4) = (P_2'P_1P_4P_3)$. These quadrics have l, l' as their common generators, therefore they have in common two other generators which meet l in Q, R and l' in Q', R' respectively. Then $(P_1P_2, QR) = -1 = (P_1'P_2', Q'R') = (P_3P_4, QR) = (P_3'P_4', Q'R')$. For, $(P_1'P_2', Q'R') = P_1(P_2, QR) = (P_2'P_1', Q'R')$ as $P_1P_1', QQ', RR', P_2P_2'$ and $P_1'P_2, P_2'P_1, QQ', RR'$ generate quadrics. Hence $(P_1P_2, QR) = -1$. Similarly $(P_3P_4, QR) = -1 = (P_3'P_4', Q'R')$.

Conversely, if Q, R and Q', R' be point-pairs on l and l' , respectively, such that they are separated harmonically by the line pairs P_1P_1', P_2P_2' and P_3P_3', P_4P_4' , then either QQ', RR' or $QR', Q'R$ will be generators of both the quadrics given above.

Now T' may be self-polar to T_2 if a pair of skew edges of T' meets a pair of skew edges of T_2 . There are then nine cases to consider.

(a) $A'D', B'C'$ cannot meet A_2D_2, B_2C_2 unless $A'D', B'C'$ coincide with A_3D_3, B_3C_3 .

(b) Let $A'B', C'D'$ meet A_2D_2, B_2C_2 . T' is self-polar to T_2 if A', B' and C', D' are separated harmonically by the lines A_2D_2, B_2C_2 whereas A_2, D_2 and B_2, C_2 by the lines $A'B', C'D'$. But A_2, D_2 and B_2, C_2 are separated harmonically by the lines A_1D_1, B_1C_1 also. Hence, by the Lemma, either A_2B_2, C_2D_2 or C_2A_2, B_2D_2 , say A_2B_2, C_2D_2 are generators of the quadric generated by $A'B', C'D', A_1D_1, B_1C_1, A_2D_2, B_2C_2, A'D', B'C'$, therefore $C'A', B'D'$ meet C_2A_2, B_2D_2 [§1].

Again A_1B_1, C_1D_1 meet $A_2B_2, C_2D_2, A_1D_1, B_1C_1$; hence they are also generators of the quadric and therefore meet $A'B', C'D'$. Then $C'A', B'D'$ meet C_1A_1, B_1D_1 . Hence $C'A', B'D'$ coincide with C_3A_3, B_3D_3 and T' is desmic with T_1, T_2 .

(c) Similar is the case when B_2C_2, A_2D_2 meet $C'A', B'D'$.

(d) Let $A'B', C'D'$ meet A_2B_2, C_2D_2 . T' is self-polar T_2 if A', B' and C', D' are separated harmonically by the lines A_2B_2, C_2D_2 whereas A_2, B_2 and C_2, D_2 by the lines $A'B', C'D'$. Hence $A'B', C'D', A_2D_2, B_2C_2$ generate a quadric and $A'B', C'D', A_1D_1, B_1C_1, AD, BC$ generate another quadric. These quadrics have in common the generators $A'B', C'D'$ and therefore they have in common two other generators which meet $A'B', C'D', A_1D_1, B_1C_1, A_2D_2, B_2C_2$. They cannot be other than A_3D_3, B_3C_3 .

Similarly it can be proved that A_3B_3, C_3D_3 meet $A'D', B'C'$ but they also meet A_3D_3, B_3C_3 ; hence they are generators of the second quadric. Now A_1B_1, C_1D_1 meet $A_3B_3, C_3D_3, A_1D_1, B_1C_1$; hence they meet $A'B', C'D'$. Hence T' coincides with T_3 contradicting the hypothesis.

Similar are the following three cases when

(e) $C'A', B'D'$ meet A_2B_2, C_2D_2 ;

(f) $C'A', B'D'$ meet C_2A_2, B_2D_2 ;

(g) $A'B', C'D'$ meet C_2A_2, B_2D_2 .

(h) Let $A'D'$, $B'C'$ meet A_2B_2 , C_2D_2 . T' is selfpolar to T_2 if A' , D' and B' , C' are separated harmonically by the lines A_2B_2 , C_2D_2 whereas A_2 , B_2 and C_2 , D_2 by the lines $A'D'$, $B'C'$. Now P_1' , Q_1' and P_2' , Q_2' are point-pairs on $B'C'$, $A'D'$ respectively and are seen to be separated harmonically by the line-pairs $A'B'$, $C'D'$ and A_2B_2 , C_2D_2 . Hence, by the Lemma, either $P_1'P_2'$, $Q_1'Q_2'$ or $P_1'Q_2'$, $P_2'Q_1'$ are generators of the quadric generated by the lines $A'B'$, $C'D'$, A_2B_2 , C_2D_2 . •

(i) Suppose $P_1'P_2'$, $Q_1'Q_2'$, i.e., A_1D_1 , B_1C_1 are the generators. A_2D_2 , B_2C_2 meet A_1D_1 , B_1C_1 , A_2B_2 , C_2D_2 and therefore meet $A'B'$, $C'D'$. The case then reduces to (b).

(ii) Suppose $P_1'Q_2'$, $P_2'Q_1'$ are the generators. Now consider another quadric generated by $P_1'Q_2'$, $P_2'Q_1'$, A_2D_2 , B_2C_2 . This quadric has in common with the former one the generators $P_1'Q_2'$, $P_2'Q_1'$ and therefore the quadrics have in common two other generators which meet $A'B'$, $C'D'$, A_2B_2 , C_2D_2 , A_2D_2 , B_2C_2 . They cannot be other than C_2A_2 , B_2D_2 . The case then reduces to (g).

(k) Similar is the case when $A'D'$, $B'C'$ meet C_2A_2 , B_2D_2 .

10. Conversely, if T , T_1 be tetrahedra mutually selfpolar, there exists a family of tetrahedra T_2 which are selfpolar to T and desmic with T_1 . The centres of perspectivity of T_1 , T_2 lie on the pair of edges of T that meet a pair of edges of T_1 . The points common to these pairs of edges lie on a pair of edges of T_2 .

Similarly there exists a family of tetrahedra T_2' which are selfpolar to T_1 and desmic with T . T_2 and T_2' have a fixed pair of edges common.

11. There exist two tetrahedra T , T_a' which are desmic with a pair of mutually selfpolar tetrahedra T , T_1 . The pairs of vertices of T_a , T_a' on each of their common edges are harmonic.

Let T , T_1 be as in §4 and Ω_i ($i=1, 2, 3, 4$) be quadrics generated by P_1Q_2 , P_2Q_1 , A_1B_1 , C_1D_1 ; P_1Q_2 , P_2Q_1 , C_1A_1 , B_1D_1 ; P_1Q_2 , P_2Q_1 , AB , CD ; P_1Q_2 , P_2Q_1 , CA , BD respectively. Then P_1Q_2 , P_2Q_1 , the other common generators of the pairs of quadrics Ω_1 , Ω_3 and Ω_2 , Ω_4 , are the edges of a tetrahedron T_a while P_1Q_2 , P_2Q_1 , the other common generators of the pairs of quadrics Ω_2 , Ω_3 and Ω_1 , Ω_4 are edges of the other tetrahedron T_a' . This follows from

§2 and the fact that T is self-conjugate for Ω_1, Ω_2 and T_1 for Ω_3, Ω_4 .*

The other part follows from the Lemma if we take P_1Q_2, P_2Q_1 as the lines l, l' and Ω_2, Ω_3 as the quadrics therein.

12. *There exists a set of five tetrahedra such that every two of them are mutually desmic, their centres of perspectivity being the points common to the remaining three. Two pairs of edges of each tetrahedron of a triad of them are generators of a quadric for which the remaining two are self-conjugate. There are ten such quadrics called fundamental.*⁷

If T_1, T_3 be tetrahedra mutually desmic, there are then three families of the tetrahedra [§9] selfpolar to T_1 and desmic with T_3 . Hence to each family there belong two tetrahedra desmic with both T_1, T_3 [§10]; but there is a tetrahedron T_2 which has its vertices at the centres of perspectivity of T_1, T_3 and belongs to each of the three families. Hence there are three tetrahedra T_i ($i=1, 2, 3$) which are desmic with both T_1, T_3 different from T_2 with which coincide the other three. Now applying the Lemma twice over, we have the first part.

The other part is clear from §9(b).

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- ³ Baker, *Principles of Geometry*, Vol. 4, p. 244; Vol. 1, Section 3.
- ⁴ Basset, *Geometry of Surfaces*, pp. 10, 11.
- ⁵ Baker, *ibid.*, Vol. 2, p. 213.
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- ⁷ Hudson, *ibid.*

* We can deduce this result from §9 directly if we speak of involutory correspondence between the families of tetrahedra T_i, T_i' as any point on one of the common edges of T_1, T_1' fixes a T_2 as well as a T_2' [§2]. T_2, T_2' are then the united elements of the involution.

ON MUTUALLY SELF-POLAR TETRAHEDRA

By

ASGHAR HAMEED

(Communicated by Prof. C. V. H. Rao)

Introduction: In this paper, an attempt has been made to generalize to space of three dimensions, the paper "On Mutually Self-Polar Triangles,"¹ which had been developed to give the critic centres of a plane cubic curve. Here, we shall show that the idea of mutual self-polarity, as applied to tetrahedra, leads us to the double points of Kummer's Quartic Surface (§23).

If from an arbitrary point A_1 , the transversal of a pair of opposite edges of a tetrahedron T , is drawn, and on this line is taken the fourth harmonic of A_1 w. r. t. the two points, where the transversal meets the pair of opposite edges, then the three such fourth harmonics, corresponding to the three pairs of opposite edges of T , constitute a plane called the polar plane of A_1 w. r. t. T .² A tetrahedron T_1 will be said to be self-polar w. r. t. T , if the polar plane w. r. t. T of any vertex of T_1 , is its opposite face.

We prove that given any tetrahedron T , there are three families, each of ∞ tetrahedra, having A_1 as a vertex, such that the tetrahedra of these three families, are each self-polar (§1), in fact mutually (§4), to T . We then prove that the eight vertices of two *m. s. p.** tetrahedra form an associated set, if and only if, the two are in fourfold perspective (§5). Then, we consider the emergence of some examples of the configuration denoted as (g_4, s_4) by German writers on the subject (§6).

This is followed by the result that if T is *m. s. p.* to T_1 , and self-conjugate w. r. t. a quadric Q , it is also *m. s. p.* to T_1' , the reciprocal of T_1 w. r. t. Q (§7). This is the generalization of Article 4(a), of the paper referred to above, and can further be generalized to space of any number of dimensions. We then prove that if two *m. s. p.* tetrahedra are inverted w. r. t. a point and a plane, the resulting two tetrahedra are also *m. s. p.* (§8). So also, when we invert w. r. t. a pair of skew lines (§8). We prove that two *m. s. p.*

* *m. s. p.* stands for "mutually self-polar."

tetrahedra are reciprocal *w. r. t.* each of eight quadrics (§9), from which we deduce that two tetrahedra in fourfold perspective are reciprocal *w. r. t.* each of sixteen quadrics (§12). We then discuss the reciprocity of two tetrahedra of the same type, but belonging to different families (§13).

From §16, we deduce that if T , T_1 and T_2 belong to a desmic system, and P be a point on any edge of one of them, say T , and T_1' be the tetrahedron in fourfold perspective with T_1 , with P as one centre of perspective, a similar definition being given for T_2' , then T_1' and T_2' also belong to a desmic system (§17).

The rest of the paper constitutes a chain in itself, as under. Starting with two *m. s. p.* tetrahedra, we deduce from §18, the general tetrahedron which is *m. s. p.* to one, and in fourfold perspective with the other (§19*d*). We then show that there are at the most five tetrahedra, every two of which are in fourfold perspective (§20). After defining a double-four of tetrahedra, these five tetrahedra are said to form an associated set, and analogous to the case of five associated lines in space of four dimensions, we get an aggregate of fifteen tetrahedra (§22). We are led to Klein's 60_{15} and Kummer's 16_6 configurations³ (§23), to which, thus, an independent manner of approach is here established.

A 12_6 configuration, not mentioned by Hudson, in his book, referred to above, also arises (§24).

§1. Let $ABCD$ be the tetrahedron of reference T , and an arbitrary point A_1 be taken as the unit point $(1, 1, 1, 1)$. Let $B_1(x_1, y_1, z_1, t_1)$ be a point in the polar plane of A_1 *w. r. t.* T , such that A_1 lies in the polar plane of B_1 . Since the polar plane of A_1 is $\Sigma x = 0$, and that of B_1 is $\Sigma \frac{x}{x_1} = 0$, we have $\Sigma x_1 = 0$, and $\Sigma \frac{1}{x_1} = 0$.

Taking $t_1 = 1$, we get,

$$x_1 + y_1 + z_1 + 1 = 0, \text{ and } \frac{1}{x_1} + \frac{1}{y_1} + \frac{1}{z_1} + 1 = 0,$$

whence

$$x_1 + y_1 = -1 - z_1, \quad x_1 y_1 = z_1,$$

leading to

$$x_1, y_1 = -1, -z_1 \text{ or } -z_1, -1.$$

Hence, B_1 is of the form $[-1, -v, v, 1]$ or $(-v, -1, v, 1)$. Taking the first form, it is easy to see that if we take a third point

C_1 , such that the polar plane of any one of the three points A_1 , B_1 and C_1 , passes through the remaining two, then C_1 must be the point $(-1, v, -v, 1)$ or the point $(1, -1, -1, 1)$. Taking C_1 to be $(-1, v, -v, 1)$, the pole of $A_1B_1C_1$ is $D_1(1, -1, -1, 1)$, and it follows that D_1 lies in the polar planes of A_1 , B_1 and C_1 . Thus $A_1B_1C_1D_1$ is self-polar to $ABCD$. It is clear that $A_1B_1C_1D_1$ represents a family of ∞' tetrahedra. There are three such families,* their vertices being given by

$$A_1(1 \ 1 \ 1 \ 1) (1 \ 1 \ 1 \ 1) (1 \ 1 \ 1 \ 1)$$

$$B_1(-1-v \ v \ 1) (-v_1-1 \ v_1 \ 1) (-v_2 \ v_2-1 \ 1)$$

$$C_1(-1 \ v-v \ 1) (v_1-1-v_1 \ 1) (v_2-v_2-1 \ 1)$$

$$D_1(1-1-1 \ 1) (-1 \ 1-1 \ 1) (-1 \ -1 \ 1 \ 1).$$

§2. The above result may also be obtained as follows:—

The points which lie in the polar plane of A_1 , and whose polar planes pass through A_1 , must lie on the curve $\Sigma x=0$, $\Sigma \frac{1}{x}=0$, i.e., on

the cubic curve $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{x+y+z}$, which lies in the plane $\Sigma x=0$,

and is referred to the triangle, whose vertices are the points, where $\Sigma x=0$, meets the lines DA , DB , DC . This cubic curve degenerates into three lines $y+z=0$, $z+x=0$ and $x+y=0$, forming a $\Delta D_1D_2D_3$. The general point on the line $y+z=0$, $\Sigma x=0$, is $B_1(-1, -v, v, 1)$. Its polar plane passes through the opposite vertex $D_1(1, -1, -1, 1)$, and meets the line $y+z=0$, $\Sigma x=0$, in the point $C_1(-1, v, -v, 1)$. It may be seen that $A_1B_1C_1D_1$ is self-polar to $ABCD$. Now, if A_1B_1LM is self-polar to $ABCD$, then the points L and M must lie not only on the cubic curve considered above, but also on the polar plane of B_1 . We have seen that C_1 and D_1 are the only possible positions of L and M . Hence the tetrahedron $A_1B_1C_1D_1$ is fixed as soon as B_1 is fixed. Thus, corresponding to the three lines $y+z=0$, $z+x=0$, and $x+y=0$ in the plane $\Sigma x=0$, we get the three families of tetrahedra of §1.

It is obvious that the tetrahedra of the three families have A_1 as a common vertex, while the ∞' tetrahedra of a single family have

* The contents of §1, are due to Prof. C. V. Hanumanta Rao.

another common vertex. A pair K of opposite edges of a tetrahedron of one of the three families meets each of two opposite edges of $ABCD$, e.g., the edges A_1D_1 and B_1C_1 , both meet both the edges AD and BC . Also, K is fixed for each family, and pairs of vertices of the tetrahedra of the corresponding family, lying on each of the two edges K , belong to an involution, whose united elements are the points where the edge meets the corresponding pair of opposite edges of $ABCD$.

It may be remarked that the polar plane *w. r. t.* T of a point, which lies on a face (not an edge) of T , is the face itself, while that of a point, which lies on an edge of T , becomes indeterminate. Hence, to get proper self-polar tetrahedra, we shall assume that *the point, which is going to be a vertex of a tetrahedron self-polar to T , does not lie on T itself.*

§3. We now give a purely geometrical method of arriving at the three families. It may first be noted that the polar plane *w. r. t.* T of a point A_1 , does not only contain the points mentioned in the introduction, but also the point which is the fourth harmonic of the point where the transversal from A_1 to a pair of opposite edges meets one of the edges, *w. r. t.* the two vertices of T lying on that edge; and the remaining five such points. Now, let $A_1L_1M_1$ be the transversal from A_1 to the pair of opposite edges AD and BC , meeting them respectively in L_1 and M_1 . Let D_1 be the fourth harmonic of A_1 *w. r. t.* L_1 and M_1 , L_2 of L_1 *w. r. t.* A and D , and M_2 of M_1 *w. r. t.* B and C . Now, if B_1 is taken to be any point on L_2M_2 , and C_1 is taken to be the fourth harmonic of B_1 *w. r. t.* L_2 and M_2 , then $A_1B_1C_1D_1$ is self-polar *w. r. t.* $ABCD$. For, the polar plane of A_1 passes through D_1 and the line L_2M_2 by what has been said above. Hence the polar plane of A_1 is $B_1C_1D_1$. Similarly for B_1 , C_1 and D_1 . This gives us one family corresponding to the pair of opposite edges AD and BC . Similarly, we get the remaining two families.

We add that *the figure consisting of the two tetrahedra $ABCD$ and $A_1B_1C_1D_1$ depends entirely on harmonic cross-ratios.*

§4. We shall now show that the phenomenon of self-polarity as between two tetrahedra is mutual. This follows at once from the remark at the end of the last article. We also give the following analytical proof:

The transformation

$$(\xi\eta\zeta\tau) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & v'-v'-1 \\ 1-v' & v'-1 \\ 1-1 & -1 & 1 \end{pmatrix} (xyzt),$$

leads effectively to a similar transformation expressing x, y, z and t in terms of ξ, η, ζ , and τ , with v' replaced by v , where $vv'=1$. Whence, $ABCD$ is self-polar to $A_1B_1C_1D_1$, and thus self-polarity between $ABCD$ and $A_1B_1C_1D_1$ is mutual.

Each vertex and the opposite face of either tetrahedron are pole and polar plane in regard to the other tetrahedron considered as a degenerate point-quartic surface, and also when considered as a degenerate class-quartic surface.

§5. We shall now show that *the eight vertices of two m. s. p. tetrahedra do not form a set of eight associated points, except in the particular case, when the two tetrahedra are in fourfold perspective.*

Consider $ABCD$ and $A_1B_1C_1D_1$. If the eight vertices of these two tetrahedra form an associated set, we must find that the expression,

$$\lambda(x+y+z+t)^2 + \mu(-x-vy+yz+t)^2 + \mu'(-x+vy-vz+t)^2 + \lambda'(x-y-z+t)^2$$

is free of product terms for some suitable coefficients λ, μ, μ' and λ' . Considering respectively the coefficients of xt, yt and zt , we have

$$\lambda - \mu - \mu' + \lambda' = 0, \lambda - v\mu + v\mu' - \lambda' = 0, \text{ and } \lambda + v\mu - v\mu' - \lambda' = 0,$$

then leading to $\lambda = \lambda' = \mu = \mu'$.

Making use of this result, and considering the coefficients of yz, zx , and xy , we have

$$1 - v^2 - v^2 + 1 = 0, 1 - v + v - 1 = 0, \text{ and } 1 + v - v - 1 = 0.$$

Hence, we must have $v = \pm 1$.

Each of these values of v gives the same tetrahedron $A_1B_1C_1D_1$. Indicate by $A'B'C'D'$ this particular tetrahedron of the family $A_1B_1C_1D_1$, so that the coordinates of its vertices, as also those of

the vertices of another tetrahedron $A''B''C''D''$ to be discussed below, are given by,

$$\begin{array}{ll} A'(1 \ 1 \ 1 \ 1) & A''(-1 \ 1 \ 1 \ 1) \\ B'(-1-1 \ 1 \ 1) & B''(1-1 \ 1 \ 1) \\ C'(-1 \ 1-1 \ 1) & C''(1 \ 1-1 \ 1) \\ D'(1-1-1 \ 1) & D''(1 \ 1 \ 1-1). \end{array}$$

Obviously, ABCD and $A'B'C'D'$ are in fourfold perspective, the four centres of perspective being the vertices of $A''B''C''D''$. This proves the result stated at the beginning of this article. It may be noted that the vertices of ABCD and $A'B'C'D'$ form the corners of a box.

The tetrahedron $A'B'C'D'$ mentioned above, can also be obtained in various other ways, as we now indicate :

(a) It is the only tetrahedron which is common to all the three families obtained in §1. In fact, it is the only tetrahedron to be common to any two of the three families.

(b) It is the tetrahedron whose three vertices, lying in the polar plane of the fourth vertex say A' , consist of the three double points of the corresponding degenerate cubic curve obtained in §2. This statement is but another form of (a).

(c) It is the only tetrahedron from amongst the three families, which is in perspective with ABCD. This result may also be stated in the form that if two tetrahedra are m. s. p. and in perspective, they are necessarily in fourfold perspective.

(d) If two m. s. p. tetrahedra are self-conjugate w. r. t. the same quadric, they are necessarily in fourfold perspective.

For, any quadric w. r. t. which ABCD is self-conjugate, must be of the form $\Sigma ax^2=0$. If $A_1B_1C_1D_1$ is also self-conjugate w. r. t. $\Sigma ax^2=0$, then the polar plane of $A_1(1, 1, 1, 1)$, which is $\Sigma ax=0$, must be identical with $B_1C_1D_1$ which is $\Sigma x=0$, giving $a=b=c=d$. Thus the quadric must take the form $\Sigma x^2=0$. Then, if the polar plane of $B_1(-1, -v, v, 1)$ w. r. t. $\Sigma x^2=0$, which is $-x-vy+ vz+t=0$, is identical with $C_1D_1A_1$ which is $-x-\frac{y}{v}+\frac{z}{v}+t=0$, we must have $v=\pm 1$, showing that $A_1B_1C_1D_1$ is identical with $A'B'C'D'$ which is self-conjugate w. r. t. $\Sigma x^2=0$.

§6. Three $(g_4.s_4)$ configurations can be obtained from ABCD and A'B'C'D', each corresponding to two pairs of opposite edges of ABCD (and A'B'C'D'); e.g., both AB and CD are met by each of A'B' and C'D', giving four points of intersection, and both AC and BD are met by each of A'C' and B'D', giving four other points of intersection. The coordinates of these eight points of intersection are:

$$\begin{aligned} & (1 \ 1 \ 0 \ 0), & (1 \ 0 \ 1 \ 0), \\ & (0 \ 0 \ 1 \ 1), & (0 \ 1 \ 0 \ 1), \\ & (-1 \ 1 \ 0 \ 0), & (-1 \ 0 \ 1 \ 0), \\ & (0 \ 0 \ 1-1), & (0 \ 1 \ 0-1). \end{aligned}$$

These eight points, and the eight points A, B, C, D, A', B', C' and D' constitute the sixteen points of a $(g_4.s_4)$ configuration. Also, the four of the eight points of intersection, which lie on AB and CD, form a tetrahedron which is in fourfold perspective with and hence *m. s. p.* to the tetrahedron formed by the four similar points on A'C' and B'D'. The cross-ratio of the four points on each of the eight lines of the $(g_4.s_4)$ configuration is harmonic.

§7. We shall now prove that *pole and polar plane w. r. t. a tetrahedron, reciprocate w. r. t. a quadric Q into polar plane and pole respectively, w. r. t. the reciprocal tetrahedron.* Let the unique secant from a point P, to a pair of opposite edges AD and BC of the tetrahedron ABCD meet them in L_1 and L_1' respectively. Let P_1 be the harmonic conjugate of P w. r. t. L_1 and L_1' . The three points of the type P_1 determine the polar plane of P w. r. t. ABCD. Let us reciprocate w. r. t. Q, denoting the reciprocal of A by a, etc. The transversal PL_1L_1' reciprocates into the joining line of the points pad and pbc. This line taken together with the line ad constitutes the plane l_1 , and taken together with the line bc, constitutes the plane l_1' . Also, p_1 is the fourth harmonic of p w. r. t. l_1 and l_1' . The three planes of the type p_1 meet in a point which is the pole of p w. r. t. the tetrahedron abcd. This point, however, is the pole w. r. t. Q of the plane determined by the three points of the type P_1 . This establishes the result stated at the beginning of this article.

It follows that if two tetrahedra, which are *m. s. p.*, be reciprocated w. r. t. a quadric, the resulting two tetrahedra are also *m. s. p.* As a particular case, we get the result enunciated in the introduction that, if a tetrahedron T is *m. s. p.* to another tetrahedron T_1 and

self-conjugate *w. r. t.* a quadric Q , then it is also *m. s. p.* to T_1' , the reciprocal of T_1 *w. r. t.* Q . It may be added that T_1 and T_1' will be of the same type in regard to T , i.e., that the same pair of opposite edges of T will meet a pair of opposite edges of each of T_1 and T_1' .

Analytically, the general quadric *w. r. t.* which $ABCD$ is self-conjugate, is $\Sigma ax^2 = 0$. The polar plane of A_1 and pole of $B_1C_1D_1$ *w. r. t.* this quadric are respectively $\Sigma \frac{x}{a} = 0$ and $\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}\right)$,

which are obviously polar plane and pole *w. r. t.* $ABCD$. Similarly, we may consider B_1 , C_1 and D_1 . Thus the reciprocal of $A_1B_1C_1D_1$ *w. r. t.* $\Sigma ax^2 = 0$, is *m. s. p.* to $ABCD$, which was to be proved.

On similar lines, the result may be generalized to space of any number of dimensions.

§8. Let P be an arbitrary point and Π an arbitrary plane. Let L be a point of general position, and suppose PL meets Π in M . Let L' be taken on PL , such that $(PLML') = k$, a constant. Then L' is said to be the *inverse* of L *w. r. t.* P and Π . It is easy to see that in such an inversion, points invert into points, lines into lines, and cross-ratios into equal cross-ratios. Thus, from the remark at the end of §3, it follows that, if two *m. s. p.* tetrahedra are inverted *w. r. t.* a point and a plane, the resulting two tetrahedra are also *m. s. p.*

Now, let l and l' be two arbitrary lines. Let their unique transversal from a point L of general position meet them respectively in M and M' . On this transversal, take L' , such that $(MLM'/L') = k$, a constant. Then L' is called the *inverse* of L *w. r. t.* l and l' . As before, we may observe that, if two *m. s. p.* tetrahedra are inverted *w. r. t.* a pair of skew lines, the resulting two tetrahedra are also *m. s. p.** This may also be deduced from the last case, for an inversion *w. r. t.* two skew lines AB and CD is equivalent to an inversion *w. r. t.* A and BCD , followed by an inversion *w. r. t.* B and CDA , the constant of inversion being the same in all the three inversions.

When k , the constant of inversion is equal to -1 , these inversions are called *harmonic*.

* This result is due to Mr. Sahib Ram Mandan.

§9. *Two m. s. p. tetrahedra, e.g., ABCD and $A_1B_1C_1D_1$, are reciprocal w. r. t. four pairs of quadrics, the two quadrics of each pair having four generators in common,* their equations in the present case being given by*

- (i) $v\{(x+t)^2 \pm 2(x-t)(y+z)\} + (y-z)^2 = 0.$
- (ii) $-v\{(x+t)^2 \pm 2(x-t)(y+z)\} + (y-z)^2 = 0.$
- (iii) $\{(y+z)^2 \pm 2(y-z)(x+t)\} + v(x-t)^2 = 0.$
- (iv) $\{(y+z)^2 \pm 2(y-z)(x+t)\} - v(x-t)^2 = 0.$

It may be seen that in these reciprocities, when A is made to correspond to A_1 , D must necessarily correspond to D_1 ; B may correspond to B_1 or C_1 , thus giving two solutions. Similarly, when A is made to correspond to B_1 , D must necessarily correspond to C_1 ; B may correspond to D_1 or A_1 . And so on, we get the eight quadrics.

The second pair of quadrics may be written as

$$(\sqrt{v}x + y - z + \sqrt{v}t)(\sqrt{v}x - y + z + \sqrt{v}t) = \pm 2v(x-t)(y+z),$$

and the four common generators are those, where the two planes of the left-hand side meet the two planes of the right-hand side; so also for every other pair of quadrics.

Let the reciprocal of a tetrahedron PQRS w. r. t. a quadric Σ be $P'Q'R'S'$, P' corresponding to P, etc. Inverting harmonically w. r. t. a pair of skew generators of Σ , suppose PQRS and $P'Q'R'S'$ are inverted respectively into $P_1Q_1R_1S_1$ and $P_1'Q_1'R_1'S_1'$. Since Σ inverts into itself, $P_1Q_1R_1S_1$ and $P_1'Q_1'R_1'S_1'$ are also reciprocal w. r. t. Σ . Since inversion is harmonic, P_1 must lie in the polar plane of P, i.e., in $Q'R'S'$, and similarly for Q_1, R_1 and S_1 . Also P' must lie in the polar plane of P_1' , which is $Q_1R_1S_1$. Hence $P_1Q_1R_1S_1$ and $P'Q'R'S'$ are mutually inscribed; similarly also $P_1'Q_1'R_1'S_1'$ and PQRS.

Thus, if ABCD and $A_1B_1C_1D_1$ invert harmonically w. r. t. a pair of skew generators of any one of the above-mentioned eight quadrics, into $A_0B_0C_0D_0$ and $A_{10}B_{10}C_{10}D_{10}$ respectively, then $A_0B_0C_0D_0$ and $A_{10}B_{10}C_{10}D_{10}$ are m. s. p. (§8); and ABCD and $A_{10}B_{10}C_{10}D_{10}$, as also $A_1B_1C_1D_1$ and $A_0B_0C_0D_0$ are mutually inscribed.

* The contents of this article are due to Mr. Sahib Ram Mandan.

§10. It may be observed that *each of the eight quadrics, mentioned in §9, has ring contact with two others, e.g.,*

$$v\{(x+t)^2 + 2(x-t)(y+z)\} + (y-z)^2 = 0$$

has ring contact with each quadric of the second pair, the planes of the rings of contact being respectively $y-z=0$, and $x+t=0$. Thus, we get eight rings of contact. *They lie by twos in four planes, namely $x \pm t=0$, $y \pm z=0$. These four planes constitute the tetrahedron whose two pairs of opposite edges are AD, BC, and A_1D_1 , B_1C_1 , i.e., the pairs of opposite edges, one from each of ABCD, and $A_1B_1C_1D_1$, which meet. The two rings, lying in any one of these four planes, have double contact, e.g., those in the plane $y-z=0$ are the two conics $(x+t)^2 \pm 2(x-t)(y+z)=0$, and these touch one another at $(0, 1, 1, 0)$ and $(1, 0, 0, -1)$. Also, if we take two coplanar rings and the four quadrics which in pairs have ring contact along them, then the plane of the rings has the same pole w. r. t. each of these four quadrics, e.g., $(0, 1, -1, 0)$ is the pole of $y-z=0$, w. r. t. each of the quadrics of pairs (i) and (ii). It may be seen that these poles and polar planes are the vertices and opposite faces of the tetrahedron formed by the four planes in which the eight rings of contact lie.*

§11. Let us apply harmonic inversion w. r. t. two skew generators common to any one of the four pairs of quadrics, mentioned in §9, and suppose ABCD and $A_1B_1C_1D_1$ invert respectively into $A_0B_0C_0D_0$ and $A_{10}B_{10}C_{10}D_{10}$. Then $A_0B_0C_0D_0$ and $A_{10}B_{10}C_{10}D_{10}$ are m. s. p. (§8), ABCD and $A_{10}B_{10}C_{10}D_{10}$, as also $A_1B_1C_1D_1$ and $A_0B_0C_0D_0$ are mutually inscribed (§9), and, as we shall now prove, ABCD is in fourfold perspective with $A_0B_0C_0D_0$, as also $A_1B_1C_1D_1$ with $A_{10}B_{10}C_{10}D_{10}$.

Consider the second pair of quadrics of §9. Two skew generators from amongst the four, common to the two quadrics of that pair, are

$$\sqrt{v}x + y - z + \sqrt{vt} = 0, \quad x - t = 0.$$

$$\sqrt{v}x - y + z + \sqrt{vt} = 0, \quad y + z = 0.$$

In each column of the following scheme, the first two items are the points where the above two generators meet their common transversal from the third item, whose inverse w. r. t. the pair of generators is the fourth item :

$$\begin{aligned}
 &(1, -\sqrt{v}, \sqrt{v}, 1)(1, -3\sqrt{v}, -\sqrt{v}, 1)(1, \sqrt{v}, 3\sqrt{v}, 1)(+1, -\sqrt{v}, +\sqrt{v}, +1) \\
 &(-3, -\sqrt{v}, \sqrt{v}, 1)(1, \sqrt{v}, -\sqrt{v}, 1)(1, \sqrt{v}, -\sqrt{v}, 1)(-1, \sqrt{v}, -\sqrt{v}, 3) \\
 &A(1, 0, 0, 0) \quad B(0, 1, 0, 0) \quad C(0, 0, 1, 0) \quad D(0, 0, 0, 1) \\
 &A_0(-1, -\sqrt{v}, \sqrt{v}, 1)B_0(1, -\sqrt{v}, -\sqrt{v}, 1)C_0(1, \sqrt{v}, \sqrt{v}, 1)D_0 \\
 &\quad \quad \quad (-1, \sqrt{v}, -\sqrt{v}, 1).
 \end{aligned}$$

The form of the coordinates of the vertices of $A_0B_0C_0D_0$ at once suggests that this tetrahedron is in fourfold perspective with ABCD. Similarly, we may consider $A_1B_1C_1D_1$ and $A_{10}B_{10}C_{10}D_{10}$. By symmetry, the result stated at the beginning of this article is seen to be true.

§12. We now come to the particular case when $v = \pm 1$, i.e., when $A_1B_1C_1D_1$ becomes $A'B'C'D'$ of §5. Since $A'B'C'D'$ belongs to all the three families, it follows from §9 that we shall get three sets, each of four pairs of quadrics, w. r. t. each of which, ABCD and $A'B'C'D'$ will be reciprocal. It will be seen that four quadrics are common to each of these three sets. Thus, *there are in all sixteen quadrics, w. r. t. each of which, two tetrahedra in fourfold perspective are reciprocal.*

Further, it follows from §11 that when $v = 1$, then the vertices of ABCD invert into the vertices of $A'B'C'D'$ in some order w. r. t. pairs of common skew generators of the pairs of quadrics (ii) and (iv) of §9, but not (i) and (iii) which four quadrics will be common to the three sets mentioned at the beginning of this article [and when $v = -1$, then w. r. t. the pairs of common skew generators of the pairs of quadrics (i) and (iii), but not (ii) and (iv), etc.]. The two pairs of quadrics (ii) and (iv) give four such pairs of skew generators. Hence, corresponding to all the three families, *we can find altogether twelve pairs of skew lines, w. r. t. each of which, the vertices of ABCD invert harmonically into the vertices of $A'B'C'D'$, in some order.*

§13. Consider the two tetrahedra $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ given by

$$\begin{array}{ll}
 A_1 (1, 1, 1, 1) & A_2 (f, g, h, 1) \\
 B_1 (-1, -v, v, 1) & B_2 (-f, -v_1g, v_1h, 1) \\
 C_1 (-1, v, -v, 1) & C_2 (-f, v_1g, -v_1h, 1) \\
 D_1 (1, -1, -1, 1) & D_2 (f, -g, -h, 1).
 \end{array}$$

These are the general tetrahedra of two different families which are of the same type in regard to ABCD. We shall show that *they are*

reciprocal w.r.t. a quadric in regard to which ABCD is self-conjugate, if and only if, at least one of the four conditions

$$v = \pm \frac{1}{v_1}, \pm v_1$$

holds; and for each condition satisfied, the number of quadrics is two. From this, we shall deduce that the two tetrahedra will be reciprocal w.r.t. each of four such quadrics, if and only if, either v and v_1 are both numerically equal to unity, or both numerically equal to

$$i (= \sqrt{-1}).$$

Now, the general quadric w.r.t. which ABCD is self-conjugate, may be put down as $\Sigma ax^2 = 0$. If A_1 is to be the pole of $B_2C_2D_2$, the quadric must assume the form

$$\frac{x^2}{f} + \frac{y^2}{g} + \frac{z^2}{h} + t^2 = 0,$$

and w.r.t. it D_1 is the pole of $A_2B_2C_2$. Now, B_1 may either be the pole of $C_2D_2A_2$ or of $D_2A_2B_2$. In the former case, the planes

$$-\frac{x}{f} - \frac{vy}{g} + \frac{vz}{h} + t = 0 \text{ and } -\frac{x}{f} - \frac{y}{v_1g} + \frac{z}{v_1h} + t = 0$$

must coincide, i.e., $v = \frac{1}{v_1}$, and then it is easy to see that C_1 is the

pole of $D_2A_2B_2$; thus when $v = \frac{1}{v_1}$, the ordered tetrahedra $A_1B_1C_1D_1$

and $A_2B_2C_2D_2$ are reciprocal w.r.t. the quadric

$$\frac{x^2}{f} + \frac{y^2}{g} + \frac{z^2}{h} + t^2 = 0,$$

and, as may be verified, $A_1B_1C_1D_1$ and $D_2C_2B_2A_2$ are reciprocal w.r.t. the quadric

$$\frac{x^2}{f} - \frac{y^2}{g} - \frac{z^2}{h} + t^2 = 0.$$

In the latter case, when B_1 is the pole of $D_2A_2B_2$, the planes

$$-\frac{x}{f} - \frac{vy}{g} + \frac{vz}{h} + t = 0$$

and

$$-\frac{x}{f} + \frac{y}{v_1g} - \frac{z}{v_1h} + t = 0$$

must coincide, i.e., $v = -\frac{1}{v_1}$, and then $A_1B_1C_1D_1$ and $A_2C_2B_2D_2$ are reciprocal w.r.t. the quadric

$$\frac{x^2}{f} + \frac{y^2}{g} + \frac{z^2}{h} + t^2 = 0,$$

and $A_1B_1C_1D_1$ and $D_2B_2C_2A_2$ are reciprocal w.r.t.

$$\frac{x^2}{f} - \frac{y^2}{g} - \frac{z^2}{h} + t^2 = 0.$$

Similarly, it may be seen that when A_1 is made to correspond to B_2 or C_2 , then we must have $v = \pm v_1$. When $v = v_1$, then $A_1B_1C_1D_1$ and $B_2A_2D_2C_2$ are reciprocal w.r.t.

$$-\frac{x^2}{f} - \frac{y^2}{vg} + \frac{z^2}{vh} + t^2 = 0,$$

and $A_1B_1C_1D_1$ and $C_2D_2A_2B_2$ are reciprocal w.r.t.

$$-\frac{x^2}{f} + \frac{y^2}{vg} - \frac{z^2}{vh} + t^2 = 0.$$

When $v = -v_1$, then $A_1B_1C_1D_1$ and $C_2A_2D_2B_2$ are reciprocal w.r.t.

$$-\frac{x^2}{f} - \frac{y^2}{vg} + \frac{z^2}{vh} + t^2 = 0,$$

and $A_1B_1C_1D_1$ and $B_2D_2A_2C_2$ are so w.r.t.

$$-\frac{x^2}{f} + \frac{y^2}{vg} - \frac{z^2}{vh} + t^2 = 0.$$

The two tetrahedra $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ will be reciprocal w.r.t. all of these four quadrics, if $\pm v = v_1 = \frac{1}{v_1}$, or $\pm v = v_1 = -\frac{1}{v_1}$.

In the former case, both v and v_1 are numerically equal to unity, and each of $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ is in fourfold perspective with ABCD. In the latter case, both v and v_1 are numerically equal to i . Since, each of these quadrics is fixed as soon as A_1 is made the pole of a face of $A_2B_2C_2D_2$, it follows that *four is the maximum number of such quadrics*.

§14. Considering the case, when v and v_1 are each numerically equal to unity, let us find the conditions that the eight vertices of $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ should form an associated set. We shall take $v = v_1 = 1$, since it does not affect the generality of the problem

before us. Let us transform the tetrahedron of reference to $A_1B_1C_1D_1$, the substitutions to be made being

$$x+y+z+t=\xi, \quad -x-y+z+t=\eta,$$

$$x-y-z+t=\tau, \quad -x+y-z+t=\zeta.$$

Let us further write,

$$k \equiv f+g+h+1, \quad k_1 \equiv f-g-h+1,$$

$$k_2 \equiv -f+g-h+1, \quad k_3 \equiv -f-g+h+1.$$

Then the coordinates of the vertices of $A_2B_2C_2D_2$, referred to $A_1B_1C_1D_1$ are given by

$$A_2(k, k_3, k_2, k_1)$$

$$B_2(k_3, k, k_1, k_2)$$

$$C_2(k_2, k_1, k, k_3)$$

$$D_2(k_1, k_2, k_3, k).$$

Thus the eight points will form an associated set, if and only if, the product terms clear out from the expression,

$$\begin{aligned} & \lambda_1(k\xi + k_3\eta + k_2\zeta + k_1\tau)^2 + \lambda_2(k_3\xi + k\eta + k_1\zeta + k_2\tau)^2 \\ & + \lambda_3(k_2\xi + k_1\eta + k\zeta + k_3\tau)^2 + \lambda_4(k_1\xi + k_2\eta + k_3\zeta + k\tau)^2. \end{aligned}$$

Hence, we must have

$$(\lambda_1 + \lambda_4)k_2k_3 + (\lambda_2 + \lambda_3)kk_1 = 0, \quad (\lambda_1 + \lambda_4)kk_1 + (\lambda_2 + \lambda_3)k_2k_3 = 0, \dots \quad (i)$$

$$(\lambda_1 + \lambda_3)k_3k_1 + (\lambda_2 + \lambda_4)kk_2 = 0, \quad (\lambda_1 + \lambda_3)kk_2 + (\lambda_2 + \lambda_4)k_3k_1 = 0, \dots \quad (ii)$$

$$(\lambda_1 + \lambda_2)k_1k_2 + (\lambda_3 + \lambda_4)kk_3 = 0, \quad (\lambda_1 + \lambda_2)kk_3 + (\lambda_3 + \lambda_4)k_1k_2 = 0. \dots \quad (iii)$$

In the general case, when no particular relation exists between the various k 's, it is clear from equations (i), that $\lambda_1 + \lambda_4 = \lambda_2 + \lambda_3 = 0$; similarly also from (ii) and (iii). This gives $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$. Hence, there is no non-zero solution for the λ 's in the general case. The particular relations between the various k 's, at least one of which must be satisfied, are six in all, namely the three pairs of the type $k_2k_3 \pm kk_1 = 0$. It can be seen that whenever any one of the three relations of the form $k_2k_3 - kk_1 = 0$ is satisfied, a non-zero solution for the λ 's is obtained, *e.g.*, in this very case, we have $\lambda_1 = -\lambda_2 = -\lambda_3 = \lambda_4$. And that, whenever none of these three relations is satisfied, the only other possibility of a non-zero solution for the λ 's lies in the

remaining three of the six possible relations, namely the three of the type $k_2k_3 + kk_1 = 0$, holding simultaneously.

The three relations of the type $k_2k_3 - kk_1 = 0$ simplify into $f = gh$, $g = hf$, and $h = fg$, and the three relations of the type $k_2k_3 + kk_1 = 0$, which are to hold simultaneously, simplify into $1 + f^2 = g^2 + h^2$, $1 + g^2 = h^2 + f^2$, and $1 + h^2 = f^2 + g^2$. These three simultaneous relations give $f^2 = g^2 = h^2 = 1$, which give only one position of $A_2B_2C_2D_2$ other than $A_1B_1C_1D_1$, and then $ABCD$, $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ belong to a desmic system. Hence, the vertices of $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ ($v = v_1 = 1$) will form a set of eight associated points, if any one of the three relations $f = gh$, $g = hf$, $h = fg$ is satisfied, or if the two tetrahedra belong to a desmic system of which $ABCD$ is the third.

§15. We may similarly consider the second case of §13, when each of v and v_1 is numerically equal to i , and determine the condition that the eight vertices of $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ should form an associated set. We take $v = v_1 = i$, and transform the tetrahedron of reference to $A_1B_1C_1D_1$ by the substitutions,

$$\begin{aligned} x + y + z + t &= \xi, & -x + iy - iz + t &= \eta \\ x - y - z + t &= \tau, & -x - iy + iz + t &= \zeta. \end{aligned}$$

Let us further write

$$\begin{aligned} k &\equiv f + g + h + 1, & k_1 &\equiv f - g - h + 1, \\ k_2 &\equiv -f + ig - ih + 1, & k_3 &\equiv -f - ig + ih + 1. \end{aligned}$$

Then the coordinates of the vertices of $A_2B_2C_2D_2$, referred to $A_1B_1C_1D_1$, are given by

$$A_2(k, k_2, k_3, k_1)$$

$$B_2(k_3, k, k_1, k_2)$$

$$C_2(k_2, k_1, k, k_3)$$

$$D_2(k_1, k_3, k_2, k).$$

Thus the eight points will form an associated set, if and only if, the product terms clear out from the expression,

$$\begin{aligned} \lambda_1(k\xi + k_2\eta + k_3\zeta + k_1\tau)^2 + \lambda_2(k_3\xi + k\eta + k_1\zeta + k_2\tau)^2 \\ + \lambda_3(k_2\xi + k_1\eta + k\xi + k_3\tau)^2 + \lambda_4(k_1\xi + k_3\eta + k_2\zeta + k\tau)^2, \end{aligned}$$

Hence, we must have

$$(\lambda_1 + \lambda_4)k_2k_3 + (\lambda_2 + \lambda_3)kk_1 = 0, \quad (\lambda_1 + \lambda_4)kk_1 + (\lambda_2 + \lambda_3)k_2k_3 = 0 \dots (i)$$

$$\lambda_1kk_3 + \lambda_2k_3k_1 + \lambda_3kk_2 + \lambda_4k_1k_2 = 0 \dots (ii)$$

$$\lambda_1kk_2 + \lambda_2kk_3 + \lambda_3k_1k_2 + \lambda_4k_3k_1 = 0 \dots (iii)$$

$$\lambda_1k_1k_2 + \lambda_2kk_2 + \lambda_3k_3k_1 + \lambda_4kk_3 = 0 \dots (iv)$$

$$\lambda_1k_3k_1 + \lambda_2k_1k_2 + \lambda_3kk_3 + \lambda_4kk_2 = 0. \dots (v)$$

The equations (i) will be satisfied, if either $\lambda_1 + \lambda_4 = \lambda_2 + \lambda_3 = 0$, or any one of the two conditions $k_2k_3 \pm kk_1 = 0$ holds. Taking $\lambda_1 + \lambda_4 = \lambda_2 + \lambda_3 = 0$, we see that (ii) and (iv) are each equivalent to

$$\lambda_1(kk_3 - k_1k_2) + \lambda_2(k_3k_1 - kk_2) = 0,$$

and (iii) and (v) are each equivalent to

$$\lambda_1(kk_2 - k_3k_1) + \lambda_2(kk_3 - k_1k_2) = 0.$$

Hence, for a non-zero solution of the λ 's, we must have

$$k_1k_2 - kk_3 = \pm i(k_3k_1 - kk_2).$$

Now,

$$k_1k_2 - kk_3 \equiv 2(1-i)hf + 2(1+i)fg + 2(i-1)g - 2(i+1)h,$$

$$\text{and} \quad k_3k_1 - kk_2 \equiv 2(1+i)hf + 2(1-i)fg - 2(i+1)g + 2(i-1)h.$$

Hence, the above two conditions amount to $fg=h$, and $hf=g$. When any of these two conditions holds, the eight vertices form an associated set.

Now, take $k_2k_3 - kk_1 = 0$. Then, from equations (i), we have $\Sigma\lambda_1 = 0$, by virtue of which (ii), (iii), (iv) and (v) add up to zero. Thus, we have to seek for a non-zero solution of the λ 's from $\Sigma\lambda_1 = 0$, and any three of (ii), ..., (v), say (ii), (iii), (iv). Hence, eliminating λ 's from $\Sigma\lambda_1 = 0$, (ii), (iii) and (iv), we see that another possibility of the eight vertices forming an associated set consists in the holding simultaneously of the equations,

$$k_2k_3 - kk_1 = 0, \text{ and } \begin{vmatrix} 1 & 1 & 1 & 1 \\ k & k_3 & k_3k_1 & k & k_2 & k_1k_2 \\ k & k_2 & k & k_3 & k_1k_2 & k_3k_1 \\ k_1k_2 & k & k_2 & k_3k_1 & k & k_3 \end{vmatrix} = 0.$$

Similarly, taking the remaining possibility $k_2k_3 + kk_1 = 0$, we may see that the only other possibility of the eight vertices forming an

associated set consists in the holding simultaneously of $k_2k_3 + kk_1 = 0$, and the above determinantal equation, when the second and third elements of the first row have been changed in sign.

§16. In this article, we shall prove that, if P be any point on any of the two lines $x=t$, $y+z=0$, and $x+t=0$, $y=z$, then the tetrahedron, whose vertices lie on the four lines PA_1 , PB_1 , PC_1 and PD_1 respectively, and which is *m. s. p.* to $A_1B_1C_1D_1$, is *m. s. p.* to the tetrahedron whose vertices lie respectively on PA , PB , PC and PD , and which is *m. s. p.* to $ABCD$.

From §5(c), it is clear that the tetrahedron, say $A_1'B_1'C_1'D_1'$, whose vertices lie respectively on the four lines $PA_1, \dots, PD_1$ and which is *m. s. p.* to $A_1B_1C_1D_1$, is the one which is in fourfold perspective with $A_1B_1C_1D_1$ with P as one centre of perspective. Hence, taking P to be the general point on $x=t$, $y+z=0$, namely $(1, g, -g, 1)$, the vertices of this tetrahedron, as also those of $A'B'C'D'$ whose vertices lie respectively on $PA, \dots, PD$, and which is *m. s. p.* to $ABCD$, can be put down as :

$$\begin{array}{ll} A'_1(0, 1-g, 1+g, 0) & A'(-1, g, -g, 1) \\ B'_1(v-g, 0, 0, v+g) & B'(1, -g, -g, 1) \\ C'_1(v+g, 0, 0, v-g) & C'(1, g, g, 1) \\ D'_1(0, 1+g, 1-g, 0) & D'(-1, -g, g, 1) \end{array}$$

We have now to show that $A_1'B_1'C_1'D_1'$ and $A'B'C'D'$ are *m. s. p.* For this, let us transform the tetrahedron of reference to $A'B'C'D'$, by means of the following relations :

$$\begin{array}{ll} -x + g'y - g'z + t = \xi, & x - g'y - g'z + t = \eta, \\ -x - g'y + g'z + t = \tau, & x + g'y + g'z + t = \zeta, \end{array}$$

where $g' = \frac{1}{g}$.

Then the coordinates of the vertices of $A_1'B_1'C_1'D_1'$ become

$$\begin{array}{ll} A'_1(-1, -g', g', 1), \\ B'_1(1, vg', vg', 1), \\ C'_1(1, -vg', -vg', 1), \\ D'_1(-1, g', -g', 1), \end{array}$$

showing that $A_1'B_1'C_1'D_1'$ and $A'B'C'D'$ are *m. s. p.*, which was to be proved. Similarly, we may take P to be the general point on the line $x+t=0, y=z$.

We may add that the lines $x=t, y+z=0$, and $x+t=0, y=z$, are a pair of opposite edges common to the tetrahedron which is in fourfold perspective with ABCD with A_1 (or D_1) as a centre of perspective, and the tetrahedron which is in fourfold perspective with ABCD, with B_1 (or C_1) as a centre of perspective. Similarly, the two lines are also a pair of common edges of the two tetrahedra which are in fourfold perspective with $A_1B_1C_1D_1$, with respectively A (or D) and B (or C) as centres of perspective.

§17. Coming to the particular case when $A_1B_1C_1D_1$ belongs to all the three families of §1, i.e., when $v=\pm 1$, we see that $A_1'B_1'C_1'D_1'$ is not only *m. s. p.* to $A'B'C'D'$ but also in fourfold perspective with it. Also, in this case, we shall have three pairs of lines, of the type $x=t, y+z=0$, and $x+t=0, y=z$, constituting the six edges of the tetrahedron formed by the four centres of perspective of ABCD and $A_1B_1C_1D_1$ ($v=\pm 1$). Hence, we have the following:

If ABCD and $A'B'C'D'$ belong to a desmic system, and a point P is taken anywhere on the edges of the third tetrahedron of the desmic system, then the tetrahedron, whose vertices lie respectively on the lines PA, ..., PD and which is in fourfold perspective with ABCD, is in fourfold perspective with the tetrahedron whose vertices lie respectively on PA', ..., PD', and which is in fourfold perspective with $A'B'C'D'$.

§18. Consider the tetrahedra ABCD and $A_1B_1C_1D_1$, and a general point P (x_1, y_1, z_1, t_1). Let $A_1'B_1'C_1'D_1'$ be the tetrahedron which is in fourfold perspective with $A_1B_1C_1D_1$, with P as one centre of perspective. Then, we shall prove in this article that, if $A_1'B_1'C_1'D_1'$ is *m. s. p.* to ABCD, the locus of P consists of the pair of opposite edges of ABCD, which is met by a pair of opposite edges of $A_1B_1C_1D_1$.

By the help of transformations of §4, or otherwise, it can be proved that the vertices of $A_1'B_1'C_1'D_1'$ are given by,

$$A_1'\{-x_1+y_1+z_1+t_1(\equiv k_1), x_1-y_1+z_1+t_1(\equiv k_2), \\ x_1+y_1-z_1-t_1(\equiv k_3), x_1+y_1+z_1-t_1(\equiv k_4)\}$$

$$B_1'\{x_1 - \frac{y_1}{v} + \frac{z_1}{v} + t_1 (\equiv l_1), -vx_1 + y_1 + z_1 + vt_1 (\equiv l_2),$$

$$vx_1 + y_1 + z_1 - vt_1 (\equiv l_3), x_1 + \frac{y_1}{v} - \frac{z_1}{v} + t_1 (\equiv l_4)\}$$

$$C_1'\{l_4, l_3, l_2, l_1\}$$

$$D_1'\{k_4, k_3, k_2, k_1\}.$$

Making A_1' the unit point, the coordinates of the vertices of $A_1'B_1'C_1'D_1'$ are given by the rows of the determinant R.

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ l_1/k_1 & l_2/k_2 & l_3/k_3 & l_4/k_4 \\ l_4/k_1 & l_3/k_2 & l_2/k_3 & l_1/k_4 \\ k_4/k_1 & k_3/k_2 & k_2/k_3 & k_1/k_4 \end{vmatrix} \quad \dots \quad R$$

Since $A_1'B_1'C_1'D_1'$ has further to be m.s.p. to ABCD, it must belong to, at least, one of the three families of §1. Let us first consider the first family. We shall have to consider three cases, namely, when the vertices B_1' , C_1' and D_1' , have, in turn, their coordinates $(1, -1, -1, 1)$; say D_1' is the point $(1, -1, -1, 1)$. Then, comparing the rows of R with those of

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & -v & v & 1 \\ -1 & v & -v & 1 \\ 1 & -1 & -1 & 1 \end{vmatrix}$$

we have the following relations:

$$l_1/k_1 + l_4/k_4 = 0 \quad \dots \quad (i)$$

$$l_2/k_2 + l_3/k_3 = 0 \quad \dots \quad (ii)$$

$$l_4/k_1 + l_1/k_4 = 0 \quad \dots \quad (iii)$$

$$l_3/k_2 + l_2/k_3 = 0 \quad \dots \quad (iv)$$

$$\frac{l_2}{k_2} \frac{l_1}{k_1} = \frac{l_2}{k_3} \frac{l_4}{k_1} \quad \dots \quad (v)$$

$$\text{and} \quad k_4/k_1 = -k_3/k_2 = -k_2/k_3 = k_1/k_4. \quad \dots \quad (vi)$$

We observe that none of the eight quantities $k_1, \dots, l_1, \dots$, can be zero

or ∞ , for then some vertices of $A_1'B_1'C_1'D_1'$ will lie on some faces of ABCD (see the remark at the end of §2). Hence, on addition,

$$(i) \text{ and } (iii) \text{ give } l_1 + l_4 = 0, \text{ or } k_1 + k_4 = 0.$$

Similarly (ii) and (iv) give $l_2 + l_3 = 0$, or $k_2 + k_3 = 0$.

Starting with $l_1 + l_4 = 0$, we get from (i), $k_1 = k_4$, and from (v), $k_2 + k_3 = 0$, whence from (ii) $l_2 = l_3$. Thus, we have the set of relations

$$l_1 + l_4 = 0, \quad l_2 = l_3, \quad k_1 = k_4, \quad \text{and} \quad k_2 + k_3 = 0,$$

which amount to

$$x_1 + t_1 = 0, \quad x_1 = t_1, \quad x_1 = t_1, \quad \text{and} \quad x_1 + t_1 = 0.$$

Hence, we get $x_1 = t_1 = 0$, which satisfy (i), ... (vi). Similarly, starting with the other possibility $k_1 + k_4 = 0$, we shall ultimately get $y_1 = z_1 = 0$, satisfying (i), ... (vi). Thus, part of the locus of P consists of the pair of lines,

$$x = t = 0, \quad y = z = 0. \quad \dots (L)$$

Now, let B_1' become the point $(1, -1, -1, 1)$. Then we shall get the relations:

$$l_4/k_1 + l_1/k_4 = 0 \quad \dots (i)$$

$$l_3/k_2 + l_2/k_3 = 0 \quad \dots (ii)$$

$$k_4/k_1 + k_1/k_4 = 0 \quad \dots (iii)$$

$$k_3/k_2 + k_2/k_3 = 0 \quad \dots (iv)$$

$$\frac{l_3}{k_2} \bigg/ \frac{l_4}{k_1} = \frac{k_2}{k_3} \bigg/ \frac{k_4}{k_1} \quad \dots (v)$$

$$l_1/k_1 = -l_2/k_2 = -l_3/k_3 = l_4/k_4. \quad \dots (vi)$$

From (iii), $k_1 = \pm ik_4$, and from (iv), $k_2 = \pm ik_3$.

Let us take $k_1 = ik_4$, and $k_2 = ik_3$. Then from (i), $l_1 = il_4$, and from (ii) $l_2 = il_3$. These four relations are respectively equivalent to

$$(i+1)x_1 + (i-1)y_1 + (i-1)z_1 - (i+1)t_1 = 0,$$

$$(i-1)x_1 + (i+1)y_1 - (i+1)z_1 + (i-1)t_1 = 0,$$

$$(i-1)x_1 + (i+1)\frac{y_1}{v} - (i+1)\frac{z_1}{v} + (i-1)t_1 = 0,$$

$$(i+1)vx_1 + (i-1)y_1 + (i-1)z_1 - (i+1)vt_1 = 0.$$

Of these, the first and the fourth are equivalent to $x_1 = t_1$, $y_1 + z_1 = 0$, and the second and third to $x_1 + t_1 = 0$, $y_1 = z_1$, except in the particular case when $v=1$. Thus, in general, the relations are inconsistent,

and do not yield any position for P. Similarly, we may discuss the combination $k_1 = -ik_4$, and $k_2 = -ik_3$, by changing the sign of i throughout in the last analysis. There remain two other combinations, namely, $k_1 = ik_4$, $k_2 = -ik_3$, and $k_1 = -ik_4$, $k_2 = ik_3$. As before, each of these two combinations will yield four simultaneous relations, which will be inconsistent unless $v = -1$. This disposes of the case when B'_1 is the point $(1, -1, -1, 1)$, and the case when C'_1 is $(1, -1, -1, 1)$ is similar to the last, being obtained from it, as is obvious from R, by a simultaneous interchange of l_1 and l_4 , and l_2 and l_3 . Thus we see that, if $A'_1B'_1C'_1D'_1$ belong to the first family of §1, then the locus of P consists of the lines (L).

Now, let us consider the second family of §1, taking D'_1 to be the point $(-1, 1, -1, 1)$. As before, we get the relations:

$$l_2/k_2 + l_4/k_4 = 0 \quad \dots (i)$$

$$l_1/k_1 + l_3/k_3 = 0 \quad \dots (ii)$$

$$l_3/k_2 + l_1/k_4 = 0 \quad \dots (iii)$$

$$l_4/k_1 + l_2/k_3 = 0 \quad \dots (iv)$$

$$\frac{l_1}{k_1} \Big/ \frac{l_4}{k_4} = \frac{l_2}{k_3} \Big/ \frac{l_1}{k_4} \quad \dots (v)$$

$$-k_4/k_1 = k_3/k_2 = -k_2/k_3 = k_1/k_4. \quad \dots (vi)$$

From (vi), we have $k_1 = \pm ik_4$, and $k_2 = \pm ik_3$.

Let $k_1 = ik_4$, then from (i) and (iv), we must have $k_3 = ik_2$. Now,

(v) may be written as $l_1^2 = \frac{k_1}{k_3} l_2 l_4 = -l_4^2$, from (iv). Thus $l_1 = \pm il_4$.

Taking $l_1 = il_4$, and noting that from (i) and (ii), $\frac{l_2}{k_2} \cdot \frac{l_1}{k_1} = \frac{l_4}{k_4} \cdot \frac{l_3}{k_3}$, we get

$l_3 = il_2$. We now have four simultaneous relations, $k_1 = ik_4$, $k_3 = ik_2$, $l_1 = il_4$, and $l_3 = il_2$, which, as before, do not yield any solution for P unless $v = -1$. Similarly, we may discuss the combinations

$k_1 = ik_4$, $l_1 = -il_4$; $k_1 = -ik_4$, $l_1 = il_4$; and $k_1 = -ik_4$, $l_1 = -il_4$.

It will be seen that they do not yield any positions for P, unless v is numerically equal to unity.

Let us now take B'_1 to be $(-1, 1, -1, 1)$. As before, we get

$$l_4/k_1 + l_2/k_3 = 0 \quad \dots (i)$$

$$l_3/k_2 + l_1/k_4 = 0 \quad \dots (ii)$$

$$k_4/k_1 + k_2/k_3 = 0 \quad \dots (iii)$$

$$k_3/k_2 + k_1/k_4 = 0 \quad \dots (iv)$$

$$\frac{l_4}{k_1} \int \frac{l_1}{k_4} = \frac{k_2}{k_3} \int \frac{k_1}{k_4} \quad \dots (v)$$

$$-l_1/k_1 = l_2/k_2 = -l_3/k_3 = l_4/k_4. \quad \dots (vi)$$

From (iii), we get $k_1k_2 + k_3k_4 = 0$, and from (vi), $l_2k_3 + l_3k_2 = 0$.

These two equations, when written in full, give, respectively,

$$x_1y_1 + z_1t_1 = 0, \text{ and } (1+v)z_1x_1 + y_1t_1 + (1-v)(x_1y_1 + z_1t_1) = 0.$$

Thus we get $x_1y_1 + z_1t_1 = 0$, and $z_1x_1 + y_1t_1 = 0$, in the general case. These two quadratic equations (representing two quadrics) give us the following (four common generators):

$$y_1 = z_1, x_1 + t_1 = 0; x_1 = t_1, y_1 + z_1 = 0; y_1 = z_1 = 0; \text{ and } x_1 = t_1 = 0.$$

Of these, $y_1 = z_1, x_1 + t_1 = 0$, make $l_1 = k_2 = k_3 = l_4 = 0$, and hence cannot give a solution for P. So also for $x_1 = t_1, y_1 + z_1 = 0$. Taking $y_1 = z_1 = 0$, we see that (iii) (iv) and (v) and two of the relations (vi) are satisfied. The third of the relations (vi) gives

$$-\frac{x_1 + t_1}{-x_1 + t_1} = \frac{-vx_1 + vt_1}{x_1 + t_1}, \text{ i.e., } (x_1 + t_1)^2 + v(x_1 - t_1)^2 = 0,$$

which is also given by (i) and (ii). Thus, we get two positions for P, namely $(\sqrt{v+i}, 0, 0, \sqrt{v-i})$, and $(\sqrt{v-i}, 0, 0, \sqrt{v+i})$. But it can be verified that for each of these two positions of P, $A_1'B_1'C_1'D_1'$ belongs to families of all the three types, i.e., it is in fourfold perspective with ABCD. Similar results are obtained by taking $x_1 = t_1 = 0$. Thus the case, when B_1' is $(-1, 1, -1, 1)$ is exhausted; the case when C_1' is $(-1, 1, -1, 1)$ is similar to the last, and can be obtained from it, by a simultaneous interchange of l_1 and l_4 , and l_2 and l_3 . It follows now that, when $A_1'B_1'C_1'D_1'$ belongs to the second family of §1, the corresponding positions of P lie on the lines (L), and then, $A_1'B_1'C_1'D_1'$ is in fourfold perspective with ABCD. Similarly also, when $A_1'B_1'C_1'D_1'$ belongs to the third family of §1. Combining all these cases, corresponding to the three families of §1, we get the result stated at the beginning of this article.

§19. In continuation of the theorem of the last article, we may add the following:—

(a) It is easy to see that, for all values of v , the lines $A_1'D_1'$ and $B_1'C_1'$ remain fixed as a pair, their respective equations being $x=t$, $y+z=0$, and $x+t=0$, $y=z$, when P lies on $x=t=0$, and $x+t=0$,

$y=z$, and $x=t$, $y+z=0$, when P lies on $y=z=0$. These two lines may be interpreted as in §16.

(b) No positions for P were obtained when $A_1'B_1'C_1'D_1'$ was considered to belong to the second (and similarly, the third) family, except when it belonged to all the three families. Hence, it follows that *two tetrahedra, each of which is m.s.p. to a third tetrahedron, cannot be in fourfold perspective, unless, as a first condition, one of them belongs to, at least, the same type w.r.t. the third, to which the other belongs.*

(c) Coming to the particular case when $v = \pm 1$, i.e., when $A_1B_1C_1D_1$ is in fourfold perspective with $ABCD$, the locus of P consists of all the six edges of $ABCD$.

(d) It may be observed that as soon as one position for P is fixed, three others are thereby determined, being the remaining three centres of perspective of $A_1B_1C_1D_1$ and $A_1'B_1'C_1'D_1'$. Thus the P 's occur in fours; let us call one such set as P_1, P_2, P_3, P_4 . Since $P_1P_2P_3P_4$ belongs to the desmic system determined by $A_1B_1C_1D_1$ and $A_1'B_1'C_1'D_1'$, it follows that no three or four of $P_1, \dots, P_4$ can lie on a line. Hence two of them lie on $x=t=0$, and the other two on $y=z=0$. Thus, g being a parameter, the general tetrahedron $P_1P_2P_3P_4$, and the corresponding tetrahedron $A_1'B_1'C_1'D_1'$ may be put down as

$$\begin{array}{ll} P_1(0, 1-g, 1+g, 0) & A_1'(1, g, -g, 1) \\ P_2(v-g, 0, 0, v+g) & B_1'(-1, -\frac{v}{g}, -\frac{v}{g}, 1) \\ P_3(v+g, 0, 0, v-g) & C_1'(-1, \frac{v}{g}, \frac{v}{g}, 1) \\ P_4(0, 1+g, 1-g, 0) & D_1'(1, -g, g, 1). \end{array}$$

The tetrahedron $A_1'B_1'C_1'D_1'$, as put down above, is then the general tetrahedron to be in fourfold perspective with $A_1B_1C_1D_1$, and m.s.p. to $ABCD$.

(e) If $A_1'B_1'C_1'D_1'$ is also in fourfold perspective with $ABCD$, we must have $g = \pm \frac{v}{g}$, i.e., $g = \pm \sqrt{v}$, $\pm i\sqrt{v}$. It is obvious that $g = \pm \sqrt{v}$ give the same tetrahedron $A_1'B_1'C_1'D_1'$, as also do $g = \pm i\sqrt{v}$. It follows that, when two tetrahedra are m.s.p., there are in general two and only two other tetrahedra, each of which is in fourfold perspective with each of the original two. For the two solutions of $A_1'B_1'C_1'D_1'$, so found, it is enough to take $g = \sqrt{v}$, $i\sqrt{v}$.

§20. Coming to the particular case when $v = \pm 1$, i.e., when $A_1B_1C_1D_1$ belongs to all the three families of §1, it follows from §19(d) that there are three types of tetrahedra each of which is m.s.p. to ABCD and in fourfold perspective with $A_1B_1C_1D_1$ ($v=1$). The coordinates of their vertices are given by

$$\begin{aligned} A_1'(1, g', -g', 1) \quad A_1''(g'', 1, -g'', 1) \quad A_1'''(g''', -g''', 1, 1) \\ B_1'\left(-1, \frac{-1}{g'}, \frac{-1}{g'}, 1\right) B_1''\left(\frac{-1}{g''}, -1, \frac{-1}{g''}, 1\right) B_1'''\left(\frac{-1}{g'''}, \frac{-1}{g'''}, -1, 1\right) \\ C_1'\left(-1, \frac{1}{g'}, \frac{1}{g'}, 1\right) C_1''\left(\frac{1}{g''}, -1, \frac{1}{g''}, 1\right) C_1'''\left(\frac{1}{g'''}, \frac{1}{g'''}, -1, 1\right) \\ D_1'(1, -g', g', 1) \quad D_1''(-g'', 1, g'', 1) \quad D_1'''(-g''', g''', 1, 1). \end{aligned}$$

When each of these three tetrahedra is also in fourfold perspective with ABCD, then, as in § 19(e), we must have each of g' , g'' and g''' numerically equal to unity or i . In the former case, i.e., when g' , g'' and g''' are each numerically equal to unity, the three tetrahedra $A_1'B_1'C_1'D_1'$, $A_1''B_1''C_1''D_1''$ and $A_1'''B_1'''C_1'''D_1'''$ coincide with the third tetrahedron of the desmic system determined by ABCD and $A_1B_1C_1D_1$ ($v=1$). In the latter case, we get three tetrahedra, each of which is in fourfold perspective with ABCD and $A_1B_1C_1D_1$ ($v=1$). Denoting these three tetrahedra, respectively, by b , c and d , the coordinates of their vertices are given by

$$\begin{array}{ccc|ccc} (1, i, -i, 1) & & (i, 1, -i, 1) & & (i, -i, 1, 1) & \\ (-1, i, i, 1) & & (i, -1, i, 1) & & (i, i, -1, 1) & \\ (-1, -i, -i, 1) & & (-i, -1, -i, 1) & & (-i, -i, -1, 1) & \\ (1, -i, i, 1) & & (-i, 1, i, 1) & & (-i, i, 1, 1) & \end{array} \left. \begin{array}{c} b \\ c \\ d \end{array} \right\}.$$

The tetrahedra ABCD and $A_1B_1C_1D_1$ ($v=1$) will henceforth be denoted by e and a respectively. It is easy to verify that every two of the tetrahedra b , c , and d are in fourfold perspective. The centres of perspective of c, d ; d, b ; and b, c are given respectively by the vertices of the tetrahedra n, m and p given below:

$$\begin{array}{ccc|ccc} (0, -1, 1, 0) & & (1, 0, -1, 0) & & (1, -1, 0, 0) & \\ (0, 1, 1, 0) & & (1, 0, 1, 0) & & (1, 1, 0, 0) & \\ (1, 0, 0, 1) & & (0, 1, 0, 1) & & (0, 0, 1, 1) & \\ (-1, 0, 0, 1) & & (0, -1, 0, 1) & & (0, 0, -1, 1) & \end{array} \left. \begin{array}{c} n \\ m \\ p \end{array} \right\}.$$

It is now clear that every two of the tetrahedra a, b, c, d and e are in fourfold perspective. Also, it may be verified that the tetrahedron formed by the centres of perspective of any two of a, b, c, d and e is not in fourfold perspective with any of the remaining three. Thus, from the manner of generation of a, b, c, d and e , it follows that *five is the maximum number of tetrahedra, every two of which are in fourfold perspective*. It is interesting to observe that *no three of these five tetrahedra belong to a desmic system*. These five tetrahedra are entirely symmetrical between themselves, and can be generated from any two of them, in the manner shown above.

We have already remarked that no three of $a, \dots, e$ belong to a desmic system. But, if we take any three of them, then *the three tetrahedra formed by the centres of perspective of pairs taken from those three, belong to a desmic system*. Thus, n, m and p which are formed by the centres of perspective of pairs taken from b, c and d , may be seen to belong to a desmic system.

§21. We shall now show that, whenever three tetrahedra are such that every two of them are in fourfold perspective, but the three do not belong to a desmic system, say the tetrahedra are e, a and b , then a unique quadric can be drawn to pass through their vertices; this quadric will pass through two pairs of opposite edges of each of the three tetrahedra e, a and b , and will have each of the remaining two, namely c and d , self conjugate *w. r. t.* it. For, consider $xt - yz = 0$, $xy + zt = 0$, and $xz + yt = 0$, which are three linearly independent quadrics passing through the vertices of e and b . Since by §5, the vertices of e and b form a set of eight associated points, the general quadric through them may be put down as

$$xt - yz + k_1(xy + zt) + k_2(xz + yt) = 0.$$

This will pass through $(1, 1, 1, 1)$ and $(1, -1, -1, 1)$, if $k_1 + k_2 = 0$, and through $(-1, -1, 1, 1)$ and $(-1, 1, -1, 1)$ if $k_1 - k_2 = 0$. Thus, it will pass through the vertices of a , if $k_1 = k_2 = 0$. Hence, $xt - yz = 0$ is the unique quadric passing through the vertices of e, a and b . The remaining parts of the statement made at the beginning of this article are easy to verify.

Since, groups of three tetrahedra can be taken from amongst a, b, c, d and e in ${}^5C_3 (= 10)$ ways, we get in all ten quadrics of the type $xt - yz = 0$. These have been called the *ten fundamental quadrics* by Hudson,⁴ who arrives at the tetrahedra and these quadrics by means of the theory of Line Geometry.

§22. Let a' , b' , c' and d' be respectively the third tetrahedra of the desmic systems determined by e , a ; e , b ; e , c ; and e , d . Then, in analogy to the double four of lines, and five lines of an associated set, in space of four dimensions, the sets a , b , c , d and a' , b' , c' , d' will here be said to constitute a *double four of tetrahedra*, and e will be said to be *associated* with a , b , c , d , as also with a' , b' , c' , d' . To pursue the analogy further, we shall show that we are ultimately led to fifteen tetrahedra, giving us in all six sets of *five associated tetrahedra* each. And, as in space of four dimensions, it will be found that each one of the fifteen tetrahedra gives rise to a double four associated with it, and that every two of the six sets of five associated tetrahedra have one and only one tetrahedron in common.

Since, a' was taken as the third tetrahedron of the desmic system constituted by a pair (namely e , a) from a , b , c , d and e , we get in all ${}^5C_2 (=10)$ tetrahedra of the type a' . These, together with a , b , c , d and e , give us the fifteen tetrahedra, we had in view. They are given in the following scheme, in which the two rows of tetrahedra in a paranthesis constitute a double four, of which the associated tetrahedron is given outside the paranthesis:

$$\begin{array}{c}
 \begin{array}{c} 1 \ 0 \ 0 \ 0 \\ 0 \ 1 \ 0 \ 0 \\ 0 \ 0 \ 1 \ 0 \\ 0 \ 0 \ 0 \ 1 \end{array} \left| \begin{array}{c} \begin{array}{c} 1 \ 1 \ 1 \ 1 \\ -1 \ -1 \ 1 \ 1 \\ -1 \ 1 \ -1 \ 1 \\ 1 \ -1 \ -1 \ 1 \\ -1 \ 1 \ 1 \ 1 \\ 1 \ -1 \ 1 \ 1 \\ 1 \ 1 \ -1 \ 1 \\ 1 \ 1 \ 1 \ -1 \end{array} \\ e \end{array} \right| \begin{array}{c} \begin{array}{c} 1 \ i \ -i \ 1 \\ -1 \ i \ i \ 1 \\ -1 \ -i \ -i \ 1 \\ 1 \ -i \ i \ 1 \\ 1 \ i \ i \ 1 \\ -1 \ -i \ i \ 1 \\ -1 \ i \ -i \ 1 \\ 1 \ -i \ -i \ 1 \end{array} \\ a \end{array} \left| \begin{array}{c} \begin{array}{c} i \ 1 \ -i \ 1 \\ i \ -1 \ i \ 1 \\ -i \ -1 \ -i \ 1 \\ -i \ 1 \ i \ 1 \\ i \ 1 \ i \ 1 \\ -i \ -1 \ i \ 1 \\ i \ -1 \ -i \ 1 \\ -i \ 1 \ -i \ 1 \end{array} \\ b \end{array} \right| \begin{array}{c} \begin{array}{c} -i \ i \ 1 \ 1 \\ i \ i \ -1 \ 1 \\ -i \ -i \ -1 \ 1 \\ i \ -i \ 1 \ 1 \\ i \ i \ 1 \ 1 \\ i \ -i \ -1 \ 1 \\ -i \ i \ -1 \ 1 \\ -i \ -i \ 1 \ 1 \end{array} \\ c \end{array} \right| \begin{array}{c} \begin{array}{c} d \\ d' \end{array} \end{array}
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{c} e \\ a' \end{array} \left| \begin{array}{c} \begin{array}{c} i \ 0 \ 0 \ 1 \\ 0 \ i \ 1 \ 0 \\ 0 \ 1 \ i \ 0 \\ 1 \ 0 \ 0 \ i \end{array} \\ a' \end{array} \right| \begin{array}{c} \begin{array}{c} 0 \ i \ 0 \ 1 \\ i \ 0 \ 1 \ 0 \\ 1 \ 0 \ i \ 0 \\ 0 \ 1 \ 0 \ i \end{array} \\ b' \end{array} \right| \begin{array}{c} \begin{array}{c} 0 \ 0 \ i \ 1 \\ 1 \ i \ 0 \ 0 \\ i \ 1 \ 0 \ 0 \\ 0 \ 0 \ 1 \ i \end{array} \\ c' \end{array} \right| \begin{array}{c} \begin{array}{c} d \\ l \end{array} \end{array}
 \end{array}$$

$$\begin{array}{c}
 \left[\begin{array}{cccc|ccc}
 & a & & e & & c & & d \\
 & & & & 1-1 & 0 & 0 & \\
 & & & & 1 & 1 & 0 & 0 \\
 & r & & b' & & & & \\
 & & & & 0 & 0 & 1 & 1 \\
 & & & & 0 & 0 & -1 & 1
 \end{array} \right] \begin{array}{l} \\ \\ p \\ \\ m \end{array}
 \end{array}$$

$$\begin{array}{c}
 \left[\begin{array}{cccc|ccc}
 & a & & b & & e & & d \\
 & & & & & & 0-1 & 1 & 0 \\
 & & & & & & 0 & 1 & 1 & 0 \\
 & q & & p & & c' & & \\
 & & & & & & 1 & 0 & 0 & 1 \\
 & & & & & & -1 & 0 & 0 & 1
 \end{array} \right] \begin{array}{l} \\ \\ n \\ \\ \\ \end{array}
 \end{array}$$

$$\begin{array}{c}
 \left[\begin{array}{cccc}
 & a & & b & & c & & e \\
 & l & & m & & n & & d'
 \end{array} \right]
 \end{array}$$

These fifteen tetrahedra, which we have denoted by $e, a, b, c, d, a', b', c', d', l, m, n, p, q$ and r , have been called the *fifteen fundamental tetrahedra* by Hudson,⁵ who arrives at them by means of the theory of Line Geometry (see also §21). The six sets of associated tetrahedra, mentioned at the beginning of this article, are now given by the rows (also by the columns) of the following scheme,⁶ which is the same as gives similar sets for lines in space of four dimensions:

	I	II	III	IV	V	VI
I	.	e	d	a	b	c
II	e	.	d'	a'	b'	c'
III	d	d'	.	l	m	n
IV	a	a'	l	.	r	q
V	b	b'	m	r	.	p
VI	c	c'	n	q	p	.

The four other tetrahedra in the row, and the four other tetrahedra in the column, of any one of the fifteen tetrahedra, constitute the double four associated with it.

It may be observed that the fifteen tetrahedra are symmetrical between themselves, and may be generated from any of the six associated sets. In each of these associated sets, two tetrahedra are real, and the rest imaginary. •

§23. We shall now consider some of the configurations arising from the vertices, edges and faces of the fifteen tetrahedra. The entire figure consists of sixty vertices, thirty edges, and sixty faces. The sixty vertices, and the sixty faces constitute Klein's 60_{15} configuration,⁷ i.e., each of the faces contains fifteen of the vertices, and each of the vertices lies in fifteen of the faces. We may, however, remark that the fifteen points, lying in any face, can be divided into a group of six points, and three groups of three points each, such that the six points of the first group lie on a conic section of one of the ten fundamental quadrics (§21), w.r.t. which the points of each of the remaining three groups form a self-conjugate triangle. For example, consider the plane $x=0$. It contains three vertices of e , and a pair of vertices of each of r, q, l, p, m and n . The six vertices arising from r, q and l lie on the conic $\Sigma x^2=0, x=0$. Each of the three self-conjugate triangles is formed by one of the three vertices of e , and a pair of vertices of p, m , or n , which lie on the opposite edge of e in the plane $x=0$. These three triangles are tabled below.

$$\begin{array}{ccc|ccc} (0 & 1 & 0 & 0) & (0 & 0 & 1 & 0) & (0 & 0 & 0 & 1) \\ (0 & 0 & 1 & 1) & (0 & 1 & 0 & 1) & (0 & 1 & 1 & 0) \\ (0 & 0 & -1 & 1) & (0 & -1 & 0 & 1) & (0 & -1 & 1 & 0) \end{array}$$

The results also hold reciprocally; e.g., the enveloping cone of $\Sigma x^2=0$ from $(1, 0, 0, 0)$ is touched by the six faces of r, q and l which pass through $(1, 0, 0, 0)$. And if the points of the above three triads are replaced by the opposite faces of the tetrahedra of which they are vertices, these faces being concurrent at $(1, 0, 0, 0)$, then every two faces of a triad are conjugate w.r.t. the enveloping cone, considered above.

Now, consider any of the fifteen double-fours of tetrahedra given by the second scheme of the last article. Then the sixteen vertices of one set of four tetrahedra and the sixteen faces of the other set constitute Kummer's 16_6 configuration,⁸ i.e., each of these sixteen faces passes through six of the sixteen vertices, and each of the sixteen

vertices lies in six of the sixteen faces. Thus the entire figure of the fifteen tetrahedra gives $15 \times 2 (=30)$ configurations of Kummer.

§24. In this article, we shall show how *the vertices and faces of some of the fifteen tetrahedra give a new 12_6 configuration*. It is known ⁷ that each of the fifteen tetrahedra belongs to four desmic systems, and that there are, in all, twenty such systems. This may also be seen from the second scheme of §22. If we consider any of the fifteen tetrahedra, and mark, in serial order, the remaining tetrahedra in its row and column, from left to right in the former case, and from top to bottom in the latter case, then the tetrahedron belongs to four desmic systems, determined by the k th tetrahedra in its row and column, $k=1, 2, 3, 4$. Since each tetrahedron belongs to four desmic systems, and a desmic system is given by three tetrahedra, the number of such systems, as already stated, is $\frac{15 \times 4}{3} = 20$.

Now, these twenty desmic systems can be grouped in ten pairs, e.g., e, a, a' and p, n, m , such that the twelve vertices of one system of a pair, and the twelve faces of the other system, form a 12_6 configuration, i.e., each of these twelve faces passes through six of the twelve vertices, and each of the twelve vertices lies in six of the twelve faces. This has not been mentioned by Hudson in his book referred to above. The manner of grouping of the twenty desmic systems is this. Take two tetrahedra of any one system. The associated set determined by them contains three other tetrahedra. These three tetrahedra, taken in pairs, give three desmic systems, of which the three third tetrahedra belong to a desmic system (§20), and this system is to be grouped with the one with which we started. Thus, starting with e, a and a' , the associated set, determined by e and a , contains b, c and d . These three give three desmic systems, $c, d; d, b$; and b, c . The third tetrahedra of these three desmic systems are respectively n, m and p , which constitute the desmic system to be grouped with e, a and a' . We may remark that, of the ten pairs, this is the only real pair.

§25. No face of e passes through any vertex of the eight tetrahedra $a, b, c, d, a', b', c', d'$, which constitute the double-four associated with e . But the remaining six, namely l, m, n, p, q and r , can be divided into three pairs, $l, p; m, q$; and n, r , such that e has a pair of opposite edges in common with the two tetrahedra of each of the three pairs. Thus, we get three triads of tetrahedra, such that the tetrahedra of each triad have a pair of opposite edges in common.⁸ Since each of

the fifteen tetrahedra gives us three such triads, and each triad arises from three tetrahedra, the number of such triads is $\frac{15 \times 3}{3} = 15$.

§26. A set of three edges of a tetrahedron, which is such that no two of them are skew, may be of two forms. Either it may consist of the three edges which concur at a vertex of the tetrahedron, or it may consist of the three edges which lie in a face of it. If we take such sets, of any forms, one from each of five associated tetrahedra, then the five sets, so obtained, are such that any edge of one set meets one and only one edge of each of the remaining four sets. This is so, because any pair of opposite edges of one of two tetrahedra in fourfold perspective meet a pair of opposite edges of the second; and each set contains one edge from each of the three pairs of opposite edges of the corresponding tetrahedron. Thus, each of the fifteen edges, given by the five sets, meets six others. These fifteen lines can be given Schläfli's notation.⁹ For example, the edges of five sets, taken respectively from e , a , b , c and d , can be denoted as follows:

$$\begin{array}{lll} x=t=0(c_{12}) & y=t=0(c_{34}) & z=t=0(c_{56}) \\ x+t=0, y+z=0(c_{35}) & y+t=0, z+x=0(c_{26}) & z+t=0, x+y=0(c_{14}) \\ x+t=0, y=z(c_{46}) & y+it=0, z=ix(c_{25}) & z=it, x=iy(c_{13}) \\ x+it=0, y+iz=0(c_{36}) & y+t=0, z=x(c_{15}) & z=it, x+iy=0(c_{24}) \\ x=it, y+iz=0(c_{45}) & y+it=0, z+ix=0(c_{16}) & z+t=0, x=y(c_{23}). \end{array}$$

It is easy to verify that two c 's intersect if, and only if, they have no common suffix, e.g., c_{12} intersects c_{34} , but not c_{13} .

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- ⁴ *Ibid.*, §24. ⁵ *ibid.*, §§23, 29.
- ⁶ Reproduced from Baker, *Ibid.*, p. 114.
- ⁷ Hudson, *ibid.*, §25. ⁸ Hudson, *ibid.*, §26.
- ⁹ See Salmon, *Analytic Geometry of Three Dimensions*, Vol. II, §534.

THE TORTUOSITY OF SUBMANIFOLDS OF A VARIETY

By

N. N. GHOSH

In a recent paper * I have considered some properties relating to the osculating and the normal subspaces of a general variety immersed in an Euclidean hyperspace and have obtained new formulæ for the inclination of two neighbouring subspaces of the same type. The object of the present paper is to continue similar investigations with regard to submanifolds of a variety including curves as a particular case. Much of what is contained in my previous paper will be made use of in this connection.

In W. Mayer's paper there is a short treatment¹ regarding submanifolds of a variety. In the present paper I have started with his line of investigation attempting at greater comprehensiveness and making good some incompleteness therein.

In the 'Intrinsic Geometry of ideal space,' Vol. I, by Forsyth, chapter VI is devoted to a *nonvectorial* treatment of submanifolds including the metrical properties connected with geodesics. In § 4 I have obtained the formulæ in much simpler way.

Let the variety containing the submanifolds be of r dimensions and let it be referred to an orthogonal system of n unit vectors ϵ_i passing through an origin O in an Euclidean n -space. A point on the r -fold variety is then given by the vector

$$A = x_i \epsilon_i \quad (i=1, 2, \dots, n)$$

where the x 's are scalar functions of r parameters $t_1, t_2, t_3, \dots, t_r$.

Fixing upon a point P_0 on the variety which corresponds to the parameters t_j , a point P_1 in its vicinity corresponding to parameters $t_j + u_j$ is determined by the vector $P_0 P_1$, i.e., by

$$A(t_1 + u_1, t_2 + u_2, \dots, t_r + u_r) - A(t_1, t_2, \dots, t_r). \quad \dots \quad (2)$$

A submanifold of q dimensions included in the r -fold variety in the neighbourhood of P_0 will then be defined by (2) where the u 's are regarded as functions of q parameters $v_1, v_2, \dots, v_q$.

* "The tortuosity of a variety"—*Bull. Cal. Math. Soc.*, 32 (1940), 51-60. We shall refer to this paper as *Tortuosity*.

Let Q_0 be a definite point on the submanifold corresponding to the parameters v_λ ($\lambda=1, 2, \dots, q$). Denoting the vector $P_0 Q_0$ by $B(v_1, v_2, \dots, v_q)$, we have from (2) by differentiation with respect to the parameters v_λ

$$\frac{\partial B}{\partial v_\lambda} = A_j(t_1 + u_1, t_2 + u_2, \dots, t_r + u_r) \frac{\partial u_j}{\partial v_\lambda} \quad (j=1, 2, \dots, r),$$

which we write in the form

$$B_\lambda = C_j \frac{\partial u_j}{\partial v_\lambda}, \quad \dots \quad (3)$$

the system of vectors C_j forming the tangent subspace of the variety at Q_0 .

From (3) by further differentiations we have

$$B_{\lambda_1 \lambda_2} = C_j \frac{\partial^2 u_j}{\partial v_{\lambda_1} \partial v_{\lambda_2}} + C_{j_1 j_2} \frac{\partial u_{j_1}}{\partial v_{\lambda_1}} \cdot \frac{\partial u_{j_2}}{\partial v_{\lambda_2}}, \quad \dots \quad (4)$$

$$\begin{aligned} B_{\lambda_1 \lambda_2 \lambda_3} = & C_j \frac{\partial^3 u_j}{\partial v_{\lambda_1} \partial v_{\lambda_2} \partial v_{\lambda_3}} + C_{j_1 j_2} \left[\frac{\partial^2 u_{j_1}}{\partial v_{\lambda_1} \partial v_{\lambda_2}} \cdot \frac{\partial u_{j_2}}{\partial v_{\lambda_3}} \right. \\ & \left. + \frac{\partial^2 u_{j_1}}{\partial v_{\lambda_2} \partial v_{\lambda_3}} \cdot \frac{\partial u_{j_2}}{\partial v_{\lambda_1}} + \frac{\partial^2 u_{j_1}}{\partial v_{\lambda_3} \partial v_{\lambda_1}} \cdot \frac{\partial u_{j_2}}{\partial v_{\lambda_2}} \right] \\ & + C_{j_1 j_2 j_3} \frac{\partial u_{j_1}}{\partial v_{\lambda_1}} \cdot \frac{\partial u_{j_2}}{\partial v_{\lambda_2}} \cdot \frac{\partial u_{j_3}}{\partial v_{\lambda_3}}, \quad \dots \quad (5) \end{aligned}$$

and so on.

It follows from the above that the osculating subspace of any order p of the submanifold at a point Q_0 is included in the osculating subspace of the same order $\dagger$ of the variety at that point. As basis for the latter we shall take the system of $\binom{r+p}{p} - 1$ vectors

$$C_j, C_{j_1 j_2}^*, \dots, C_{j_1 j_2 \dots j_p}^{*/p-1}$$

defining the tangent and the successive normal subspaces of the variety.

$\dagger$ As p increases this statement undergoes modification because the enveloping Euclidean space is of limited dimension.

2. Successive normal subspaces of the submanifold

Let us denote the scalar determinant corresponding to the tangent subspace at Q_0 of the submanifold by

$$\Delta_{B_q} = \begin{vmatrix} B_1 & B_2 & \dots & B_q \\ B_1 & B_2 & \dots & B_q \end{vmatrix}, \quad \dots \quad (6)$$

then the conjugate system $B_{(\lambda)}$ is given by

$$B_{(\lambda)} = B_{\mu} \begin{vmatrix} B_{\mu} \\ B_{\lambda} \end{vmatrix}_c / \Delta_{B_q} \quad (\mu=1, 2, \dots, q), \quad \dots \quad (7)$$

where the suffix c attached to the minor of (6) indicates that its complementary is to be taken.

The first normal subspace of the submanifold at Q_0 is then defined by

$$B_{\lambda_1 \lambda_2}^* = B_{\lambda_1 \lambda_2} - B_{\lambda} \begin{vmatrix} B_{(\lambda)} \\ B_{\lambda_1 \lambda_2} \end{vmatrix}.$$

Since*

$$B_{\lambda_1 \lambda_2} = C_j \begin{vmatrix} C_{(j)} \\ B_{\lambda_1 \lambda_2} \end{vmatrix} + C_{j_1 j_2}^* \begin{vmatrix} C_{(j_1 j_2)}^* \\ B_{\lambda_1 \lambda_2} \end{vmatrix},$$

the above may be expressed in the form

$$B_{\lambda_1 \lambda_2}^* = C_{j/1} \begin{vmatrix} C_{(j)} \\ B_{\lambda_1 \lambda_2} \end{vmatrix} + C_{j_1 j_2}^* \begin{vmatrix} C_{(j_1 j_2)}^* \\ B_{\lambda_1 \lambda_2} \end{vmatrix}, \quad \dots \quad (8)$$

where

$$C_{j/1} = C_j - B_{\lambda} \begin{vmatrix} B_{(\lambda)} \\ C_j \end{vmatrix}. \quad \dots \quad (9)$$

The system (9) does not consist of r linearly independent vectors. They all lie on the tangent subspace of the variety at Q_0 and are orthogonal to the tangent subspace of the submanifold at Q_0 . Thus only $r-q$ of them are, in general, linearly independent. These may be called the *residual vectors of the first order*. Denoting them by

* Tortuosity, formula (4'1).

v_{ξ_1} and the conjugate system by $v_{(\xi_1)}$, where ξ_1 runs over the values $q+1, q+2, \dots, r$, we can express (8) in the form

$$B_{\lambda_1 \lambda_2}^* = v_{\xi_1} \left\{ B_{\lambda_1 \lambda_2}^{v(\xi_1)} \right\} + C_{j_1 j_2}^* \left\{ B_{\lambda_1 \lambda_2}^{C_{j_1 j_2}^*} \right\} \quad \dots \quad (10)$$

It must be noticed from (4) that the coefficient of $C_{j_1 j_2}^*$ in the above is equivalent to

$$\frac{\partial u_{j_1}}{\partial v_{\lambda_1}} \cdot \frac{\partial u_{j_2}}{\partial v_{\lambda_2}}.$$

Let the system of vectors conjugate † to $B_{\lambda_1 \lambda_2}^*$ be denoted by $B_{(\lambda_1 \lambda_2)}^*$, then the *second normal subspace* of the submanifold at Q_0 is determined by the vectors

$$B_{\lambda_1 \lambda_2 \lambda_3}^{**} = B_{\lambda_1 \lambda_2 \lambda_3} - B_{\mu_1 \mu_2}^* \left\{ B_{\lambda_1 \lambda_2 \lambda_3}^{B_{\mu_1 \mu_2}^*} \right\} - B_{\mu} \left\{ B_{\lambda_1 \lambda_2 \lambda_3}^{B_{\mu}} \right\} \quad \dots \quad (11)$$

Introducing the systems of vectors

$$C_{j/2} = C_j - B_{\mu_1 \mu_2}^* \left\{ C_j^{B_{\mu_1 \mu_2}^*} \right\} - B_{\mu} \left\{ C_j^{B_{\mu}} \right\}, \quad \dots \quad (12)$$

$$C_{j_1 j_2 / 2}^* = C_{j_1 j_2}^* - B_{\mu_1 \mu_2}^* \left\{ C_{j_1 j_2}^{B_{\mu_1 \mu_2}^*} \right\}, \quad \dots \quad (13)$$

we may express (11) in the form

$$\begin{aligned} B_{\lambda_1 \lambda_2 \lambda_3}^{**} = & C_{j/2} \left\{ C_{j/2} \right\}_{B_{\lambda_1 \lambda_2 \lambda_3}} + C_{j_1 j_2 / 2}^* \left\{ C_{j_1 j_2 / 2}^* \right\}_{B_{\lambda_1 \lambda_2 \lambda_3}} \\ & + C_{j_1 j_2 j_3}^{**} \left\{ C_{j_1 j_2 j_3}^{**} \right\}_{B_{\lambda_1 \lambda_2 \lambda_3}}. \quad \dots \quad (14) \end{aligned}$$

Now, the vectors of the systems (12) and (13) belong to the osculating subspace of the second order of the variety at Q_0 and are

† Defined in accordance with formula (27), *Tortuosity*.

orthogonal to the osculating subspace of the second order of the submanifold at the point. Hence only $\binom{r+2}{2} - \binom{q+2}{2}$ of the vectors will be linearly independent. These may be called the *residual vectors of the second order*. Denoting them by v_{ξ_2} and the conjugate system by $v_{(\xi_2)}$, where ξ_2 ranges over the values

$$\binom{q+2}{2}, \binom{q+2}{2} + 1, \dots, \binom{r+2}{2} - 1,$$

we can express (14) in the reduced form

$$B_{\lambda_1 \lambda_2 \lambda_3}^{**} = v_{\xi_2} \left\{ v_{(\xi_2)} \right\}_{B_{\lambda_1 \lambda_2 \lambda_3}} + C_{j_1 j_2 j_3}^{**} \left\{ C_{(j_1 j_2 j_3)}^{**} \right\}_{B_{\lambda_1 \lambda_2 \lambda_3}} \dots \quad (15)$$

Generally, the $(p-1)^{\text{th}}$ normal subspace of the submanifold at Q_0 is determined by the vectors

$$B_{\lambda_1 \lambda_2 \dots \lambda_p}^{*/p-1} = B_{\lambda_1 \lambda_2 \dots \lambda_p} - B_{\mu_1 \mu_2 \dots \mu_{p-1}}^{*/p-2} \left\{ B_{(\mu_1 \mu_2 \dots \mu_{p-1})}^{*/p-2} \right\}_{B_{\lambda_1 \lambda_2 \dots \lambda_p}} - \dots \\ \dots - B_{\mu} \left\{ B_{(\mu)} \right\}_{B_{\lambda_1 \lambda_2 \dots \lambda_p}} \dots \quad (16)$$

which may be reduced to the form

$$B_{\lambda_1 \lambda_2 \dots \lambda_p}^{*/p-1} = v_{\xi_{p-1}} \left\{ v_{(\xi_{p-1})} \right\}_{B_{\lambda_1 \lambda_2 \dots \lambda_p}} + C_{j_1 j_2 \dots j_p}^{*/p-1} \left\{ C_{(j_1 j_2 \dots j_p)}^{*/p-1} \right\}_{B_{\lambda_1 \lambda_2 \dots \lambda_p}} \dots \quad (17)$$

where $v_{\xi_{p-1}}$ denotes the *residual vectors of $(p-1)^{\text{th}}$ order*, ξ_{p-1} ranging over the values

$$\binom{q+p-1}{p-1}, \binom{q+p-1}{p-1} + 1, \dots, \binom{r+p-1}{p-1} - 1.$$

In the osculating subspace of p^{th} order of the submanifold at Q_0 we shall take the system of

$$\binom{q+p}{p} - 1 \text{ vectors } B_{\lambda}, B_{\lambda_1 \lambda_2 \dots \lambda_p}^{*/p-1}$$

as forming a *basis*. All properties depending on these fundamental vectors will follow as in my paper *Tortuosity*.

Thus, the angle $\delta\bar{\theta}_p$ between two infinitely near osculating subspaces of p^{th} order of the submanifold at Q_0 is given by †

$$(\delta\bar{\theta}_p)^2 = \delta_{\nu_{\lambda_p}} \delta_{\nu_{\mu_p}} \left\{ \begin{matrix} B^{*/p} \\ \lambda\lambda_1 \dots \lambda_p \end{matrix} \right\} \left\{ \begin{matrix} B^{*/p-1} \\ (\lambda\lambda_1 \dots \lambda_{p-1}) \end{matrix} \right\} \left\{ \begin{matrix} B^{*/p} \\ \mu\mu_1 \dots \mu_p \end{matrix} \right\} \left\{ \begin{matrix} B^{*/p-1} \\ (\mu\mu_1 \dots \mu_{p-1}) \end{matrix} \right\}, \dots \quad (18)$$

which may be expressed in a form involving also fundamental quantities of the variety by applying (17).

Similarly, the generalized Frenet formulæ ‡ corresponding to the submanifold will follow.

3. Curves on a variety

When the u 's are functions of a single parameter v , the vector (2) defines a curve on the variety (1) passing through points in the neighbourhood of P_0 . Let the point Q_0 correspond to a definite value v of the parameter, then denoting the vector P_0Q_0 by $B(v)$, we have by successive differentiations

$$B' = C_j \frac{du_j}{dv}, \dots \quad (19)$$

$$B'' = C_j \frac{d^2u_j}{dv^2} + C_{j_1j_2} \frac{du_{j_1}}{dv} \cdot \frac{du_{j_2}}{dv}, \dots \quad (20)$$

$$B''' = C_j \frac{d^3u_j}{dv^3} + 3C_{j_1j_2} \frac{d^2u_{j_1}}{dv^2} \cdot \frac{du_{j_2}}{dv} + C_{j_1j_2j_3} \frac{du_{j_1}}{dv} \cdot \frac{du_{j_2}}{dv} \cdot \frac{du_{j_3}}{dv}, \dots \quad (21)$$

and so on.

The osculating p -space at the point Q_0 is determined by the set of p vectors B' , B'' , $B''' \dots B^{(p)}$ and is contained in the osculating subspace of the p^{th} order of the variety at Q_0 . All properties of the curve based on these *fundamental vectors* $B^{(p)}$, whose maximum number is n , will follow as in one of my previous papers entitled ² 'New methods in the geometry of hyperspace.'

Let us denote the scalar determinant

$$\begin{vmatrix} B' & B'' & \dots & B^{(p)} \\ B' & B'' & \dots & B^{(p)} \end{vmatrix} \dots \quad (22)$$

† *Tortuosity*, formula (3.9).

‡ *Tortuosity*, formula (4.4).

by $\Delta_{B^{(p)}}$ then the normal v_{p-1} of the $(p-1)$ th rank is expressible in the form

$$\Delta_{B^{(p-1)}v_{p-1}} = C_j \begin{pmatrix} C_{(j)} \\ B^{(p)} \end{pmatrix} \Delta_{B^{(p)}} + C_{j_1 j_2} \begin{pmatrix} C_{(j_1 j_2)}^* \\ B^{(p)} \end{pmatrix} \Delta_{B^{(p)}} + \dots + C_{j_1 j_2 \dots j_p}^{*/p-1} \begin{pmatrix} C_{(j_1 j_2 \dots j_p)}^* \\ B^{(p)} \end{pmatrix} \Delta_{B^{(p)}} \dots \quad (23)$$

where the symbol $\begin{pmatrix} \lambda \\ B^{(p)} \end{pmatrix} \Delta_{B^{(p)}}$ denotes the scalar determinant $\Delta_{B^{(p)}}$

in which the element $B^{(p)}$ in one of the rows is replaced by λ .

If $\delta\theta_{p'}$ denotes the angle between two consecutive osculating p -spaces at the point Q_0 , then

$$\left(\frac{d\theta_{p'}}{dv} \right)^2 = \left\{ \begin{matrix} v_p \\ v_p \end{matrix} \right\} / \left\{ \begin{matrix} v_{p-1} \\ v_{p-1} \end{matrix} \right\} \quad \dots \quad (24)$$

where

$$\left\{ \begin{matrix} v_p \\ v_p \end{matrix} \right\} = \Delta_{B^{(p+1)}} / \Delta_{B^{(p)}}.$$

4. Geodesics on the variety and the submanifold

If the parameter v be changed to s defined by

$$s = \int \begin{Bmatrix} B' \\ B' \end{Bmatrix} dv,$$

and the fundamental vectors of the curve be denoted by $\beta^{(p)}$, we have then

$$\begin{Bmatrix} \beta' \\ \beta' \end{Bmatrix} = 1, \quad \begin{Bmatrix} \beta'' \\ \beta' \end{Bmatrix} = 0, \quad \text{and so on.}$$

A geodesic through Q_0 referred to the parameter s is a curve such that

$$\begin{Bmatrix} C_{(j)} \\ \beta'' \end{Bmatrix} = 0. \quad \dots \quad (25)$$

Referring to (20) the above may be put in the standard form

$$\frac{d}{ds} \begin{Bmatrix} C_{(j)} \\ \beta' \end{Bmatrix} + \begin{Bmatrix} C_{j_1 j_2} \\ C_{(j)} \end{Bmatrix} \begin{Bmatrix} C_{(j_1)} \\ \beta' \end{Bmatrix} \begin{Bmatrix} C_{(j_2)} \\ \beta' \end{Bmatrix} = 0, \quad \dots \quad (26)$$

where it must be remembered

$$\left\{ \begin{matrix} C_{(j)} \\ \beta' \end{matrix} \right\} = \frac{du_j}{ds}.$$

It follows immediately that for a geodesic

$$\beta'' = C_{j_1 j_2}^* \left\{ \begin{matrix} C_{(j_1)} \\ \beta' \end{matrix} \right\} \left\{ \begin{matrix} C_{(j_2)} \\ \beta' \end{matrix} \right\}. \quad \dots (27)$$

Differentiating the above and simplifying by means of (26) we have †

$$\beta''' = \tau_{j_1 j_2 j_3} \left\{ \begin{matrix} C_{(j_1)} \\ \beta' \end{matrix} \right\} \left\{ \begin{matrix} C_{(j_2)} \\ \beta' \end{matrix} \right\} \left\{ \begin{matrix} C_{(j_3)} \\ \beta' \end{matrix} \right\}, \quad \dots (28)$$

where

$$\tau_{j_1 j_2 j_3} = \frac{1}{3} P \left[\frac{\partial C_{j_1 j_2}^*}{\partial t_{j_3}} - 2 C_{j_1 j_2}^* \left\{ \begin{matrix} C_{(j_2 j_3)} \\ C_{(j)} \end{matrix} \right\} \right]. \quad \dots (29)$$

In the above P indicates that the sum of all terms obtained by cyclic permutations of the letters j_1, j_2, j_3 are to be taken.

Thus the fundamental vectors of the geodesic on a variety are determined when β' is assigned at the point.

Let the unit tangent vector β' also belong to the submanifold defined in (2). We have then

$$\beta' = B_{\lambda} \left\{ \begin{matrix} B_{(\lambda)} \\ \beta' \end{matrix} \right\}. \quad \dots (30)$$

The vector

$$\bar{\beta}'' = B_{\lambda_1 \lambda_2}^* \left\{ \begin{matrix} B_{(\lambda_1)} \\ \beta' \end{matrix} \right\} \left\{ \begin{matrix} B_{(\lambda_2)} \\ \beta' \end{matrix} \right\} \quad \dots (31)$$

corresponds to β'' in (27).

Referring to (10) the above becomes

$$\bar{\beta}'' = v_{\xi_1} \left\{ \begin{matrix} v_{(\xi_1)} \\ \bar{\beta}'' \end{matrix} \right\} + \beta''. \quad \dots (32)$$

Hence ‡

$$\left\{ \begin{matrix} \bar{\beta}'' \\ \beta'' \end{matrix} \right\} = \left\{ \begin{matrix} \beta'' \\ \beta'' \end{matrix} \right\}, \quad \dots (33)$$

† Usually, the successive differential coefficients of u , are thus expressed.

‡ Cf. Forsyth—*Intrinsic geometry of ideal space*, Vol. I, p. 182.

$$\begin{Bmatrix} \bar{\beta}'' \\ \bar{\beta}'' \end{Bmatrix} = \begin{Bmatrix} \beta'' \\ \beta'' \end{Bmatrix} + \begin{Bmatrix} \nu \xi_1 \\ \bar{\beta}'' \end{Bmatrix} \begin{Bmatrix} \nu(\xi_1) \\ \bar{\beta}'' \end{Bmatrix}. \quad \dots (34)$$

Let an elementary length δs be measured along the two geodesics. The tangent vectors at those points are given by $\beta'(s + \delta s)$ and $\bar{\beta}'(s + \delta s)$. If the angle between these vectors be denoted by $\delta\omega$, then

$$\begin{aligned} \sin^2 \delta\omega &= \frac{\begin{Bmatrix} \bar{\beta}'(s + \delta s) & \beta'(s + \delta s) \end{Bmatrix}}{\begin{Bmatrix} \bar{\beta}'(s + \delta s) & \beta'(s + \delta s) \end{Bmatrix}} \\ &= \delta s^2 \left[\frac{\begin{Bmatrix} \bar{\beta}'' & \beta' \end{Bmatrix}}{\begin{Bmatrix} \bar{\beta}'' & \beta' \end{Bmatrix}} + \frac{\begin{Bmatrix} \beta' & \beta'' \end{Bmatrix}}{\begin{Bmatrix} \beta' & \beta'' \end{Bmatrix}} + 2 \frac{\begin{Bmatrix} \bar{\beta}'' & \beta' \end{Bmatrix}}{\begin{Bmatrix} \beta' & \beta'' \end{Bmatrix}} \right] + \dots \end{aligned}$$

$$\text{Hence} \quad \left(\frac{d\omega}{ds} \right)^2 = \frac{\begin{Bmatrix} \bar{\beta}'' \end{Bmatrix}}{\begin{Bmatrix} \bar{\beta}'' \end{Bmatrix}} - \frac{\begin{Bmatrix} \beta'' \end{Bmatrix}}{\begin{Bmatrix} \beta'' \end{Bmatrix}}. \quad \dots (35)$$

If ψ denote the angle between β'' and $\bar{\beta}''$, it may be proved that

$$\cos^2 \psi = \frac{\begin{Bmatrix} \beta'' \end{Bmatrix}}{\begin{Bmatrix} \beta'' \end{Bmatrix}} \bigg/ \frac{\begin{Bmatrix} \bar{\beta}'' \end{Bmatrix}}{\begin{Bmatrix} \bar{\beta}'' \end{Bmatrix}}. \quad \dots (36)$$

References

- ¹ *Trans. Amer. Math. Soc.*, Vol. 38 (1936), pp. 297-300.
- ² *Bull. Cal. Math. Soc.*, Vol. 30 (1938), pp. 73-84.

ANNUAL REPORT OF THE CALCUTTA MATHEMATICAL SOCIETY FOR THE YEAR 1941

The Council of the Calcutta Mathematical Society has the pleasure to submit the following report on the general concerns of the Society for the year 1941.

The Council of the Society during the year under review was constituted as follows :—

President :—Prof. F. W. Levi, Dr. Phil.

Vice-Presidents :—Prof. N. R. Sen, D.Sc., Ph.D. (up to 31.1.41 and again from 1.8.41.)

Dr. M. R. Siddiqi, M.A., Ph.D.

Dr. H. D. Bagchi, M.A., Ph.D.

Mr. B. M. Sen, M.A., M.Sc.

Dr. N. M. Basu, D.Sc.

Treasurer :—Mr. N. C. Ray, M.A.

Secretary :—Mr. S. K. Chakrabarty, M.Sc. (up to 31.1.41 and again from 1.8.41.)

Prof. N. R. Sen, D.Sc., Ph.D. (from 1.2.41. to 31.7.41.)

Editorial Secretary :—

Dr. J. Ghosh, M.A., Ph.D. (up to 25.7.41.)

Dr. B. S. Ray, M.Sc., Ph.D. (from 26.7.41.)

Other Members of the Council :—

Mr. B. N. Mukherjee, M.A.

Mr. S. Gupta, M.Sc.

Mr. D. N. Sen, M.A.

Dr. B. S. Ray, M.Sc., Ph.D.

Dr. C. N. Srinivasiengar, D.Sc.

Dr. N. G. Shabde, D.Sc.

Dr. Gorakh Prasad, D.Sc.

Dr. R. N. Sen, M.A., Ph.D.

Mr. R. P. Mukherjee, M.A., B.L.

Prof. A. C. Banerjee, M.A.

Prof. C. V. H. Rao, M.A.

Mr. B. B. Sen, M.Sc.

The Council of the Society held 5 meetings during the year. There were altogether 6 meetings of the Society during the year in which 20 papers were read.

One symposium was held during the year 1941, on "Teaching of Mathematics in different Universities and Boards in India." Mr. B. M. Sen opened the discussion which was based on a comparative study of the syllabuses in different courses of the various Universities and Boards in India.

During the year under review 5 new members were elected. The Society lost 4 of its members by death, so that at the end of the year the Society had 26 Honorary Members, 14 Life Members and 78 Ordinary Members on its roll. The number of subscribers to the Bulletin of the Society at the end of the year was 44.

At a meeting of the Council held on the 25th July, 1941, the following gentlemen were elected to the Board of Editors:—

S. N. Bose, F. W. Levi, C. V. H. Rao, N. R. Sen, M. R. Siddiqi and B. S. Ray (*Secretary*).

The following gentlemen were appointed as Editorial Collaborators:—

A. C. Banerjee, N. M. Basu, R. C. Bose, P. N. Dasgupta, J. Ghosh, S. C. Mitra, V. V. Narlikar, B. M. Sen, R. N. Sen, N. G. Shabde, C. N. Srinivasiengar, P. L. Srivastava.

During the year, 4 numbers of the Bulletin of the Society were published, viz., Vol. 32, Nos. 3 and 4, and Vol. 33, Nos. 1, 2, so that at the close of the year the Bulletin was in arrear by 2 numbers.

During the year the Society entered into 3 new exchange relations. The total number of Societies, Libraries, etc., to whom the Society sends its Bulletin for favour of exchange stood at 109 at the end of the year. The number of libraries etc., to whom the Society sends its Bulletin on gratis was 3.

The financial condition of the Society improved during the year. The total surplus that has been added to the credit of the Society, during the year under review, is Rs. 221-13-11. The report of the auditors will show in detail the income and expenditure incurred by the Society during the year.

RECEIPTS AND EXPENDITURES OF THE CALCUTTA MATHEMATICAL SOCIETY FOR THE YEAR
ENDING 31st DECEMBER, 1941

ANNUAL REPORT

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Receipts		Expenditures	
Rs. as. p.		Rs. as. p.	
(9,168 1 4)		(1,987 4 6)	
I. Opening Balance :—		I. Expenses during the year :—	
(1) Cash Certificates of the total face value of Rs. 2000 on its valuations on 1. 1. 41.	1,912 8 0	(1) Establishment	...
(2) P. O. Savings Bank	2,959 8 7	(2) Meetings	...
(3) K. K. G. P. Fund	1,843 3 0	(3) Books and Journals (including charges for book-binding)	...
(4) With Bengal Central Bank	2,451 5 3	(4) Bulletin :—	...
(5) With Secretary	1 8 6	(a) Blocks etc.	...
		(b) Postage	...
		(c) Printing and Stationery	...
		(d) Postage	...
		(e) Bank charges	...
		(f) Deposit with P. O. Savings Bank	...
		(g) Miscellaneous (including conveyance)	...
II. Receipts during the year :—	(2,209 2 5)		
(1) Subscriptions	594 0 0		
(2) Admission fee	30 0 0		
(3) Sale proceeds	416 3 3		
(4) Interest received :—			
(i) P. O. Savings Bank	24 5 2	(1) Cash Certificates of the total face value of Rs. 2000 on its present valuation	...
(ii) B. C. Bank	8 10 0	(2) P. O. Savings Bank	...
(iii) Cash Certificates	87 8 0	(3) K. K. G. P. Fund	...
(iv) K. K. G. P. Fund	48 8 0	(4) Bengal Central Bank	...
(5) Withdrawal from B. C. Bank	1,000 0 0	(5) With Secretary	...
Total	11,877 3 9	Total	11,877 3 9

K. N. Chakravarti
A. O. Chowdhury
Auditors

K. N. Chakravarti

A. C. Chowdhury

Auditors